1 Basic filter

In Durbin and Koopmans (2012)

$$\nu_t = y_t - Z_t a_t \quad \text{(DK2012 4.13)}$$
$$F_t = Z_t P_t Z_t' + H_t \quad \text{(DK2012 4.16)}$$
$$a_{t|t} = a_t + P_t Z_t' F_t^{-1} \nu_t \quad \text{(DK2012 4.17)}$$
$$P_{t|t} = P_t - P_t Z_t' F_t^{-1} Z_t P_t \quad \text{(DK2012 4.18)}$$
$$K_t = T_t P_t Z_t' F_t^{-1} \quad \text{(DK2012 4.22)}$$
$$a_{t+1} = T_t a_t + K_t \nu_t \quad \text{(DK2012 4.21)}$$
$$P_{t+1} = T_t P_t (T_t - K_t Z_t)' + R_t Q_t R_t' \quad \text{(DK2012 4.23)}$$

Our in place algorithm

$$\nu_t = y_t - c_t - Z_t a_t$$
$$ZP = Z_t P_t$$
$$F_t = ZP Z_t' + H_t$$
$$iF\nu_t = F_t^{-1} \nu_t$$
$$\tilde{K}_t = F_t^{-1} ZP \quad \text{alternative } K$$
$$a_{t|t} = a_t + \tilde{K}_t' \nu_t)$$
$$P_{t|t} = P_t - \tilde{K}_t' (ZP)$$
$$a_{t+1} = d_t + T_t a_{t|t}$$
$$P_{t+1} = T_t P_{t|t} T_t + R_t Q_t R_t'$$

2 Diffuse filter

In Durbin and Koopmans (2012)

$$\nu_t^{(0)} = y_t - Z_t a_t^{(0)} \quad \text{(DK2012 p. 128)}$$
$$F_{\infty,t} = Z_t P_{\infty,t} Z_t' \quad \text{(DK2012 5.7)}$$
$$F_{\star,t} = Z_t P_{\star,t} Z_t' + H_t \quad \text{(DK2012 5.7)}$$
$$M_{\infty,t} = P_{\infty,t} Z_t' \quad \text{(DK2012 5.7)}$$
$$M_{\star,t} = P_{\star,t} Z_t' \quad \text{(DK2012 5.7)}$$

When $F^{-1}_{\infty,t}$ is regular

$$
\begin{aligned}
F^{(1)}_t &= F^{-1}_{\infty,t} && \text{(DK2012 5.10)}\\
F^{(2)}_t &= -F^{-1}_{\infty,t}F_{\star,t}F^{-1}_{\infty,t} && \text{(DK2012 5.10)}\\
K^{(0)}_t &= T_tM_{\infty,t}F^{(1)}_t && \text{(DK2012 5.12)}\\
K^{(1)}_t &= T_tM_{\star,t}F^{(1)}_t + T_tM_{\infty,t}F^{(2)}_t && \text{(DK2012 5.12)}\\
L^{(0}_t &= T_t - K^{(0)}_tZ_t && \text{(DK2012 5.12)}\\
L^{(1}_t &= -K^{(1)}_tZ_t && \text{(DK2012 5.12)}\\
a^{(0)}_{t|t} &= a^{(0)}_t + M_{\infty,t}F^{(1)}_t\nu^{(0)}_t\\
P_{\infty,t|t} &= P_{\infty,t} - P_{\infty,t}Z'_tF^{(1)}_tM'_{\infty,t}\\
P_{\star,t|t} &= P_{\star,t} - P_{\star,t}Z'_tF^{(1)}_tZ_tP_{\infty,t} - P_{\infty,t}Z'_t(F^{(1)}Z_tP_{\star,t} + F^{(2)}Z_tP_{\infty})\\
a^{(0)}_{t+1} &= T_ta^{(0)}_t + K^{(0)}_t\nu^{(0)}_t && \text{(DK2012 5.13)}\\
P_{\infty,t+1} &= T_tP_{\infty,t}L^{(0)'}_t && \text{(DK2012 5.14)}\\
P_{\star,t+1} &= T_tP_{\infty,t}L^{(1)'}_t + T_tP_{\star,t}L^{(0)'}_t + R_tQ_tR'_t && \text{(DK2012 5.14)}
\end{aligned}
$$

When $F^{-1}_{\infty,t} = \mathbf{0}$

$$
\begin{aligned}
K^{(0)}_t &= T_tM_{\star,t}F^{-1}_{\star,t} && \text{(DK2012 5.15)}\\
L^{(0}_t &= T_t - K^{(0)}_tZ_t && \text{(DK2012 5.12)}\\
L^{(1}_t &= -K^{(1)}_tZ_t && \text{(DK2012 5.12)}\\
a^{(0)}_{t|t} &= a^{(0)}_t + M_{\star,t}F^{-1}_{\star,t}\nu^{(0)}_t\\
P_{\infty,t|t} &= P_{\infty,t}\\
P_{\star,t|t} &= P_{\star,t} - P_{\star,t}Z'_tF^{-1}_{\star,t}Z_tP_{\star,t}\\
a^{(0)}_{t+1} &= T_ta^{(0)}_t + K^{(0)}_t\nu^{(0)}_t && \text{(DK2012 p. 129)}\\
P_{\infty,t+1} &= T_tP_{\infty,t}T'_t && \text{(DK2012 5.14)}\\
P_{\star,t+1} &= T_tP_{\star,t}L^{(0)'}_t + R_tQ_tR'_t && \text{(DK2012 5.17).}
\end{aligned}
$$

When $F^{-1}_{\infty,t}$ is singular but different from zero, one uses a univariate step.

The diffuse filter is used only for few iterations at the beginning of the computation of the filter. For some of the arrays we use the same one that will be used for the rest of the computation. Our in place algorithm is

$$\begin{aligned}
\nu_t &= y_t - c_t - Z_t a_t \\
F_{\infty,t} &= Z_t P_{\infty,t} Z_t' \\
F_{\star,t} &= Z_t P_{\star,t} Z_t' + H_t \\
ZP_{\infty} &= Z_t P_{\infty,t} \\
ZP_{\star} &= Z_t P_{\star,t}
\end{aligned}$$

When $F_{\infty,t}^{-1}$ is regular

$$\begin{aligned}
\tilde{K}_{\infty,t} &= F_t^{(1)}(ZP_{\infty}) \\
\tilde{K}_{\star,t} &= F_t^{(1)}((ZP_{\star}) + F_{\star,t} K_{\infty,t}) \\
a_{t|t} &= a_t + K_{\infty,t}' \nu_t \\
P_{\infty,t|t} &= P_{\infty,t} - \tilde{K}_{\infty,t}'(ZP_{\infty}) \\
P_{\star,t|t} &= P_{\star,t} - (ZP_{\star})' \tilde{K}_{\infty,t} - (ZP_{\infty})' \tilde{K}_{\star,t} \\
a_{t+1} &= d_t + T_t a_{t|t} \\
P_{\infty,t+1} &= T_t P_{\infty,t} T_t' \\
P_{\star,t+1} &= T_t P_{\star,t|t} T_t' + R_t Q_t R_t'
\end{aligned}$$

When $F_{\infty,t}^{-1} = \mathbf{0}$

$$\begin{aligned}
K_{\infty,t}^{(0)} &= T_t M_{\star,t} F_{\star,t}^{-1} && \text{(DK2012 5.15)} \\
a_{t|t} &= a_t + K_{\infty,t}^{-1} \nu_t^{(0)} \\
P_{\infty,t|t} &= P_{\infty,t} \\
P_{\star,t|t} &= P_{\star,t} - P_{\star,t} Z_t' F_{\star,t}^{-1} Z_t P_{\star,t} \\
a_{t+1}^{(0)} &= T_t a_t^{(0)} + K_t^{(0)} \nu_t^{(0)} && \text{(DK2012 p. 129)} \\
P_{\infty,t+1} &= T_t P_{\infty,t} T_t' && \text{(DK2012 5.14)} \\
P_{\star,t+1} &= T_t P_{\star,t} L_t^{(0)'} + R_t Q_t R_t' && \text{(DK2012 5.17).}
\end{aligned}$$

3 Basic smoother

$$
\begin{aligned}
L_t &= T_t - K_t Z_t && \text{(DK2012 p. 87)} \\
r_{t-1} &= Z_t' F_t^{-1} \nu_t + L_t' r_t && \text{(DK2012 4.38)} \\
\hat{a}_t &= a_t + P_t r_{t-1} && \text{(DK2012 4.35)} \\
N_{t-1} &= Z_t' F_t^{-1} Z_t + L_t' N_t L_t && \text{(DK2012 4.42)} \\
V_t &= P_t - P_t N_{t-1} P_t && \text{(DK2012 4.44)} \\
u_t &= F_t^{-1} \nu_t - K_t' r_t && \text{(DK2012 4.59)} \\
\hat{\epsilon}_t &= H_t u_t && \text{(DK2012 4.58)} \\
D_t &= F_t^{-1} + K_t' N_t K_t && \text{(DK2012 4.66)} \\
\mathrm{Var}(\epsilon_t | Y_n) &= H_t - H_t D_t H_t && \text{(DK2012 4.65)} \\
\hat{\eta}_t &= Q_t' R_t' r_t && \text{(DK2012 4.63)} \\
\mathrm{Var}(\eta_t | Y_n) &= Q_t - Q_t R_t' N_t R_t Q_t && \text{(DK2012 4.68)}
\end{aligned}
$$

In place basic smoother

$$
\begin{aligned}
K_t &= T \tilde{K}_t' \\
L_t &= T_t - K_t Z_t \\
r_{t-1} &= Z_t' (iF\nu)_t + L_t' r_t \\
\hat{a}_t &= a_t + P_t r_{t-1} \\
N_{t-1} &= Z_t' (iFZ)_t + L_t' N_t L_t \\
V_t &= P_t - P_t N_{t-1} P_t \\
\hat{\epsilon}_t &= H_t ((iF\nu)_t - K' r_t) \\
D_t &= F_t^{-1} + K_t' N_t K_t \\
(V\epsilon)_t &= H_t - H_t D_t H_t \\
\hat{\eta}_t &= Q_t R_t' r_t \\
(V\eta)_t &= Q_t - Q_t R_t' N_t R_t Q_t
\end{aligned}
$$

4 Diffuse smoother

$$L_t^{(0)} = T_t - K_t^{(0)} Z_t \tag{DK2012 5.12}$$

$$L_t^{(1)} = -K_t^{(1)} Z_t \tag{DK2012 5.12}$$

$$r_{t-1}^{(0)} = L_t^{(0)'} r_t^{(0)} \tag{DK2012 5.21}$$

$$r_{t-1}^{(1)} = Z_t' F^{(1)} \nu_t^{(0)} + L_t^{(0)'} r_t^{(1)} + L_t^{(1)'} r_t^{(0)} \tag{DK2012 5.21}$$

$$\hat{a}_t = a_t^{(0)} + P_{\star,t} r_{t-1}^{(0)} + P_{\infty,t} r_{t-1}^{(1)} \tag{DK2012 5.23}$$

$$N_{t-1}^{(0)} = L_t^{(0)'} N^{(0)} L_t^{(0)} \tag{DK2012 5.26}$$

$$N_{t-1}^{(1)} = Z_t' F_t^{(1)} Z_t + L_t^{(0)'} N^{(1)} L_t^{(0)} + L_t^{(1)'} N^{(0)} L_t^{(0)} \tag{DK2012 5.29}$$

$$\begin{aligned} N_{t-1}^{(2)} &= Z_t' F_t^{(2)} Z_t + L_t^{(0)'} N^{(2)} L_t^{(0)} + L_t^{(0)'} N^{(1)} L_t^{(1)} + L_t^{(1)'} N^{(1)} L_t^{(0)} \\ &\quad L_t^{(1)'} N^{(0)} L_t^{(1)} \end{aligned} \tag{DK2012 5.29}$$

$$\begin{aligned} V_t &= P_{\star,t} - P_{\star,t} N_{t-1}^{(0)} P_{\star,t} - (P_{\star,t} N_{t-1}^{(1)} P_{\infty,t})' - P_{\infty,t} N_{t-1}^{(1)} P_{\star,t} \\ &\quad - P_{\infty,t} N_{t-1}^{(2)} P_{\infty,t} \end{aligned} \tag{DK2012 5.28}$$

$$\hat{\epsilon}_t = -H_t K_t^{(0)} r_t^{(0)} \tag{DK2012 p.135}$$

$$\hat{\eta}_t = Q_t R_t' r_t^{(0)} \tag{DK2012 p.135}$$

$$\mathrm{Var}(\epsilon_t | Y_n) = H_t - H_t K_t^{(0)} N^{(0)} K_t^{(0)} \tag{DK2012 p.135}$$

$$\mathrm{Var}(\eta_t | Y_n) = Q_t - Q_t R_t' N_t^{(0)} R_t Q_t \tag{DK2012 p.135}$$

In place diffuse smoother

$$
\begin{aligned}
K_{\infty,t} &= T\tilde{K}'_{\infty,t} \\
K_t &= T\tilde{K}'_t \\
L0 &= T_t - K_{\infty,t}Z_t \\
L &= -K_t Z_t \\
r0 &= L0' r0\_1 \\
r1 &= Z'_t F^{(1)}\nu_t + (L0)' r1\_1 + L r0\_1 \\
ahat_t &= a^{(0)}_t + P_{\star,t} r0 + P_{\infty,t} r1 \\
N0 &= (L0)'(N0)(L0) \\
N1 &= Z'_t iF Z_t + (L0)'(N1\_1)(L0) + L(N0)L0 \\
N2 &= Z_t F^{-1}_{\infty,t} F_{\star,t} F^{-1}_{\infty,t} Z_t + (L0)'(N2\_1)(L0) + L^{(0)'}_t N^{(1)} L^{(1)}_t + L^{(1)'}_t N^{(1)} L^{(0)}_t \\
&\quad L^{(1)'}_t N^{(0)} L^{(1)}_t \\
V_t &= P_{\star,t} - P_{\star,t} N^{(0)}_{t-1} P_{\star,t} - (P_{\star,t} N^{(1)}_{t-1} P_{\infty,t})' - P_{\infty,t} N^{(1)}_{t-1} P_{\star,t} \\
&\quad - P_{\infty,t} N^{(2)}_{t-1} P_{\infty,t} \\
\hat{\epsilon}_t &= -H_t K^{(0)}_t r^{(0)}_t \\
\hat{\eta}_t &= Q_t R'_t r^{(0)}_t \\
\mathrm{Var}(\epsilon_t|Y_n) &= H_t - H_t K^{(0)}_t N^{(0)} K^{(0)}_t \\
\mathrm{Var}(\eta_t|Y_n) &= Q_t - Q_t R'_t N^{(0)}_t R_t Q_t
\end{aligned}
$$

5 Univariate smoother step

Initialization

$$
\begin{aligned}
r_{t,p_t} &= r_t \\
N_{t,p_t} &= N_t
\end{aligned}
$$

For $i = p_{t-1}, \ldots, 0$, if $|F_{t,i}| > 0$,

$$
\begin{aligned}
r_{t-1,i-1} &= Z'_{t,i} F^{-1}_{t,i} \nu_{t,i} + L'_{t,i} r_{t,i} && \text{(DK2012 6.15)} \\
N_{t-1,i-1} &= Z'_{t,i} F^{-1}_{t,i} Z_{t,i} + L'_{t,i} N_{t,i} L_{t,i} && \text{(DK2012 6.15)} \\
\hat{\epsilon}_{t,i} \sigma^2_{t,i} F^{-1}_{t,i} &(\nu_{t,i} - K'_{t,i} r_{t,i} && \text{(DK2012 p. 157)} \\
\mathrm{Var}(\hat{\epsilon}_{t,i}) &= \sigma^4_{t,i} F^{-2}_{t,i} (F_{t,i} + K'_{t,i} N_{t,i} K_{t,i}) && \text{(DK2012 p. 157)}
\end{aligned}
$$

if $F_{t,i} = 0$

$$
\begin{aligned}
r_{t-1,i-1} &= r_{t,i} \\
N_{t-1,i-1} &= N_{t,i}
\end{aligned}
$$

and

$$
\begin{aligned}
r_{t-1,p_{t-1}} &= T'_{t-1} r_{t,0} && \text{(DK2012 6.15)}\\
N_{t-1,p_{t-1}} &= T'_{t-1} N_{t,0} T_{t-1} && \text{(DK2012 6.15)}\\
r_{t-1} &= r_{t-1,p_{t-1}}\\
N_{t-1} &= N_{t-1,p_{t-1}}
\end{aligned}
$$

6 Univariate diffuse smoother step

Initialization

$$
\begin{aligned}
r_{t,p_t} &= r_t\\
N_{t,p_t} &= N_t
\end{aligned}
$$

For $i = p_{t-1}, \ldots, 0$, if $|F_{t,i}| > 0$,

$$
\begin{aligned}
r0_{t-1,i-1} &= L'_{\infty,i} r0_{t,i}\\
r1_{t-1,i-1} &= Z'_{t,i} F^{-1}_{t,i} \nu_{t,i} + L'_{\infty,t,i} r0_{t,i} + L'_{0,t,i} r1_{t,i}\\
N^{(0)}_{t-1,i-1} &= L'_{\infty,t,i} N^{(0)}_{t,i} L_{\infty,t,i} && \text{(DK2012 5.26)}\\
N^{(1)}_{t-1,i-1} &= Z'_{t,i} F^{(1)}_{t,i} Z_{t,i} + L'_{\infty,t,i} N^{(0)} L_t + L_{\infty,t,i} N^{(1)} L_{0,t,i} && \text{(DK2012 5.29)}\\
N^{(2)}_{t-1,-1} &= Z'_{t,i} F^{(2)}_{t,i} Z_{t,i} + L^{(0)'}_{0,t,i} N^{(2)}_{t,i} L^{(0)}_{0,t,i} + L^{(0)'}_{t,i} N^{(1)}_{t,i} L^{(1)}_{t,i} + L^{(1)'}_{t,i} N^{(1)}_{t,i} L^{(0)}_{t,i}\\
&\quad L^{(1)'}_{t,i} N^{(0)}_{t,i} L^{(1)}_{t,i} && \text{(DK2012 5.29)}
\end{aligned}
$$

if $F_{t,i} = 0$

$$
\begin{aligned}
r_{t-1,i-1} &= r_{t,i}\\
N_{t-1,i-1} &= N_{t,i}
\end{aligned}
$$

and

$$
\begin{aligned}
r_{t-1,p_{t-1}} &= T'_{t-1} r_{t,0} && \text{(DK2012 6.15)}\\
N_{t-1,p_{t-1}} &= T'_{t-1} N_{t,0} T_{t-1} && \text{(DK2012 6.15)}\\
r_{t-1} &= r_{t-1,p_{t-1}}\\
N_{t-1} &= N_{t-1,p_{t-1}}
\end{aligned}
$$