AULA 10
INTRODUÇÃO AOS MÉTODOS ESPECTRAIS
PTC 5725 (27/11/2025)

Shen, J., Tang, T., & Wang, L.-L. (2011). Spectral Methods: Algorithms, Analysis and Applications (Vol. 41). Springer. https://doi.org/10.1007/978-3-540-71041-7

Site do Shen – files
https://blogs.ntu.edu.sg/wanglilian/book/

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Equação da onda

$$\frac{\partial^2\psi}{\partial t^2}=\frac{1}{c^2}\Delta\psi \quad \text{, pondo-se } c=1 \quad \Rightarrow \frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial x^2}$$

Discretização: Leap-frog

$$U_{k+1}=2U_k-U_{k-1}+\Delta U\cdot dt^2$$

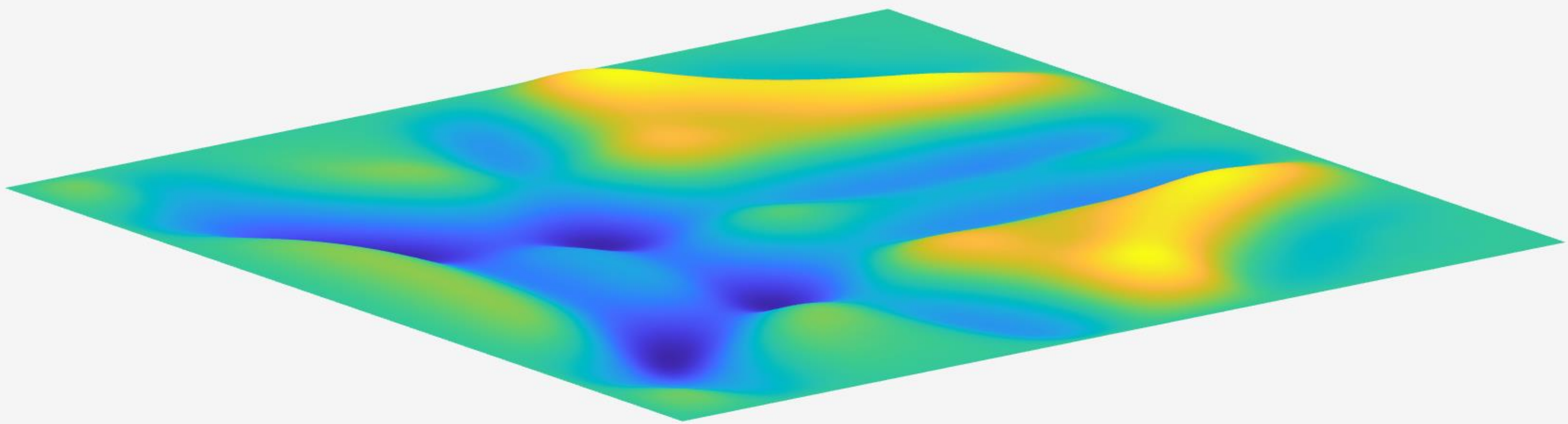
$$U(x,y,0)=e^{-40\left[(x-0.4)^2+y^2\right]}.\left(1-x^2\right)\left(1-y^2\right) \quad \text{e} \quad U_t\left(x,y,0\right)=0$$

Lembrar: $\Delta x_{\text{mín}}=1-\cos\left(\frac{\pi}{n}\right)=2.\frac{1-\cos\left(\frac{\pi}{n}\right)}{2}$

$\Delta x_{\text{mín}}=2\sin^2\left(\frac{\pi}{2n}\right)\approx\frac{\pi^2}{2n^2}\approx\frac{5}{n^2}$ $\qquad v.\Delta t<\frac{5}{n^2}$

Como $v=c=1\Rightarrow \quad \Delta t<\frac{5}{n^2}$

```matlab
% Script to simulate a travelling wave - N ~ 20
function  Wave2D_Dirich(n)

dt = 4/n^2;% max is pi^2/(2*N^2);

xs = -cos( (0:n).'*pi/n); ys = xs;
D = Generalized_Diff_Mat(xs); D2 = D^2; %Dx = Dy
[xx,yy] = meshgrid(xs,ys);
U = exp(-40*((xx-0.4).^2 + yy.^2)).*(1-xx.^2).*(1-yy.^2);
%%
Uold = U;

%%
kmax = round(3/dt);
%% Data for spectral plot
xp = linspace(-1,1,101).'; yp = xp; [xxx,yyy] = meshgrid(xp,yp);
        MI = MI_Bary(xs,xp);
%%
for k = 0:kmax

    Uyy = D2*U;  Uxx = U*D2.';

%% Advancing in time
    Unew = 2*U - Uold + dt^2*(Uxx+Uyy);
        Unew([1,n+1],:) = 0; Unew(:,[1,n+1]) = 0;% Dirichlet
                    Uold = U; U = Unew;
%% Spectral plot

Uplot = (MI*U)*MI.';


    surf(xxx,yyy,Uplot);
    axis([-1 1 -1 1 -1 1]);
    shading interp;
    axis off
    set(gcf,'position',[0 200 1400 800]);
    pause(0.1)
end

```



```
% Script to simulate a travelling wave - N ~ 20
function  Wave2D_Neum(n)

dt = 4/n^2;% max is pi^2/(2*N^2);

xs = -cos( (0:n).'*pi/n); ys = xs;
D = Generalized_Diff_Mat(xs);
[xx,yy] = meshgrid(xs,ys);
U = exp(-40*((xx-0.4).^2 + yy.^2)).*(1-xx.^2).*(1-yy.^2);
%%
Uold = U;

%%
kmax = round(3/dt);
%% Data for spectral plot
xp = linspace(-1,1,101).'; yp = xp; [xxx,yyy] = meshgrid(xp,yp);
        MI = MI_Bary(xs,xp);

%% Neumann
        D1n = D; D1n([1,n+1],:) = 0;
%%
for k = 0:kmax

    Uy = D1n*U; Uyy = D*Uy;  Ux = U*D1n.'; Uxx = Ux*D.';

%% Advancing in time
    Unew = 2*U - Uold + dt^2*(Uxx+Uyy);
%        Unew([1,n+1],:) = 0; Unew(:,[1,n+1]) = 0; %Only for Dirichlet
                    Uold = U; U = Unew;
%% Spectral plot

Uplot = MI*U*MI.';


    surf(xxx,yyy,Uplot);
    axis([-1 1 -1 1 -1 1]);
    shading interp;
    axis off
    set(gcf,'position',[0 200 1400 800]);
    pause(0.1)


end

```

Produto de Kronecker

If **A** is an $m \times n$ matrix and **B** is a $p \times q$ matrix, then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $pm \times qn$ block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1\times 0 & 1\times 5 & 2\times 0 & 2\times 5 \\ 1\times 6 & 1\times 7 & 2\times 6 & 2\times 7 \\ 3\times 0 & 3\times 5 & 4\times 0 & 4\times 5 \\ 3\times 6 & 3\times 7 & 4\times 6 & 4\times 7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}.$$

Matlab: $A \otimes B = \mathrm{kron}(A, B);$

Propriedades

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C},$$

$$(\mathbf{B} + \mathbf{C}) \otimes \mathbf{A} = \mathbf{B} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{A},$$

$$(k\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}),$$

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}),$$

$$\mathbf{A} \otimes \mathbf{0} = \mathbf{0} \otimes \mathbf{A} = \mathbf{0},$$

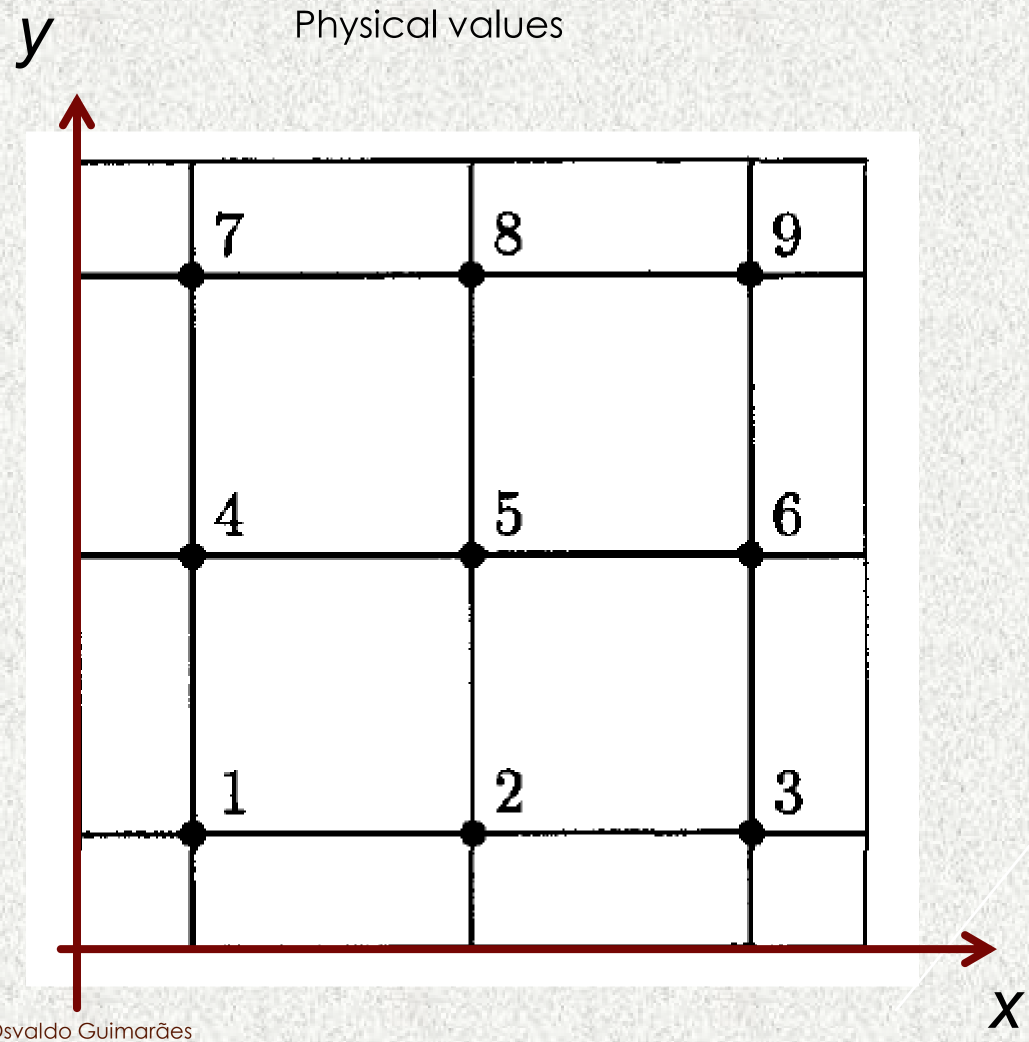
$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}).$$

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.$$

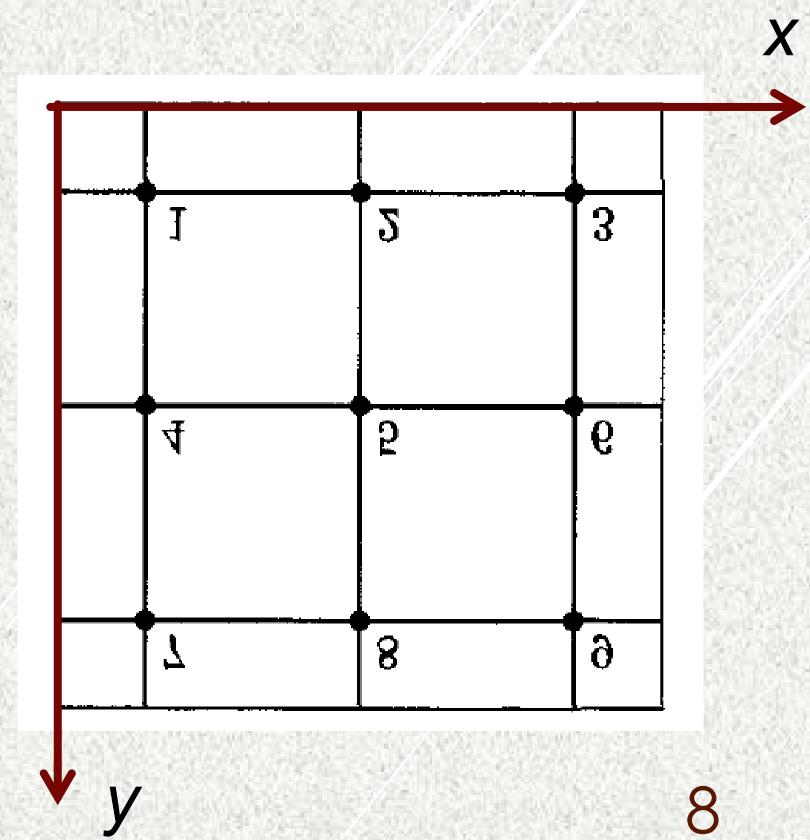
$$\boxed{\mathrm{vec}(AXB) = \left[B^T \otimes A\right] \cdot \mathrm{vec}(X)}$$

Ordem lexográfica

Physical values



Matrix position



Como aplicar a diferenciação matricial para uma matriz *M* do grid da função?

Os valores da função em y são representados nas colunas.

Assim, $\dfrac{\partial F(x,y)}{\partial y} = D_y \cdot M.$

Os valores da função em x são representados nas linhas.

Assim, $\dfrac{\partial F(x,y)}{\partial x} = \left(D_x \cdot M^T\right)^T = M \cdot D_x^T$

Considere: $\Delta F = f\left(x, y\right) \quad \left(x, y\right) \in \left[-1,1\right]^2 \subset \mathbb{R}^2$

Discretizado espectralmente, o problema fica:

$$D_y^2 F + F\left(D_x^2\right)^T = f$$

Equação de Sylvester: $AX + XB = C$

Modificar p/ que F seja a matriz central

Aplicar somente para os pontos internos, já que

$F\left(\partial\Omega\right) = 0$

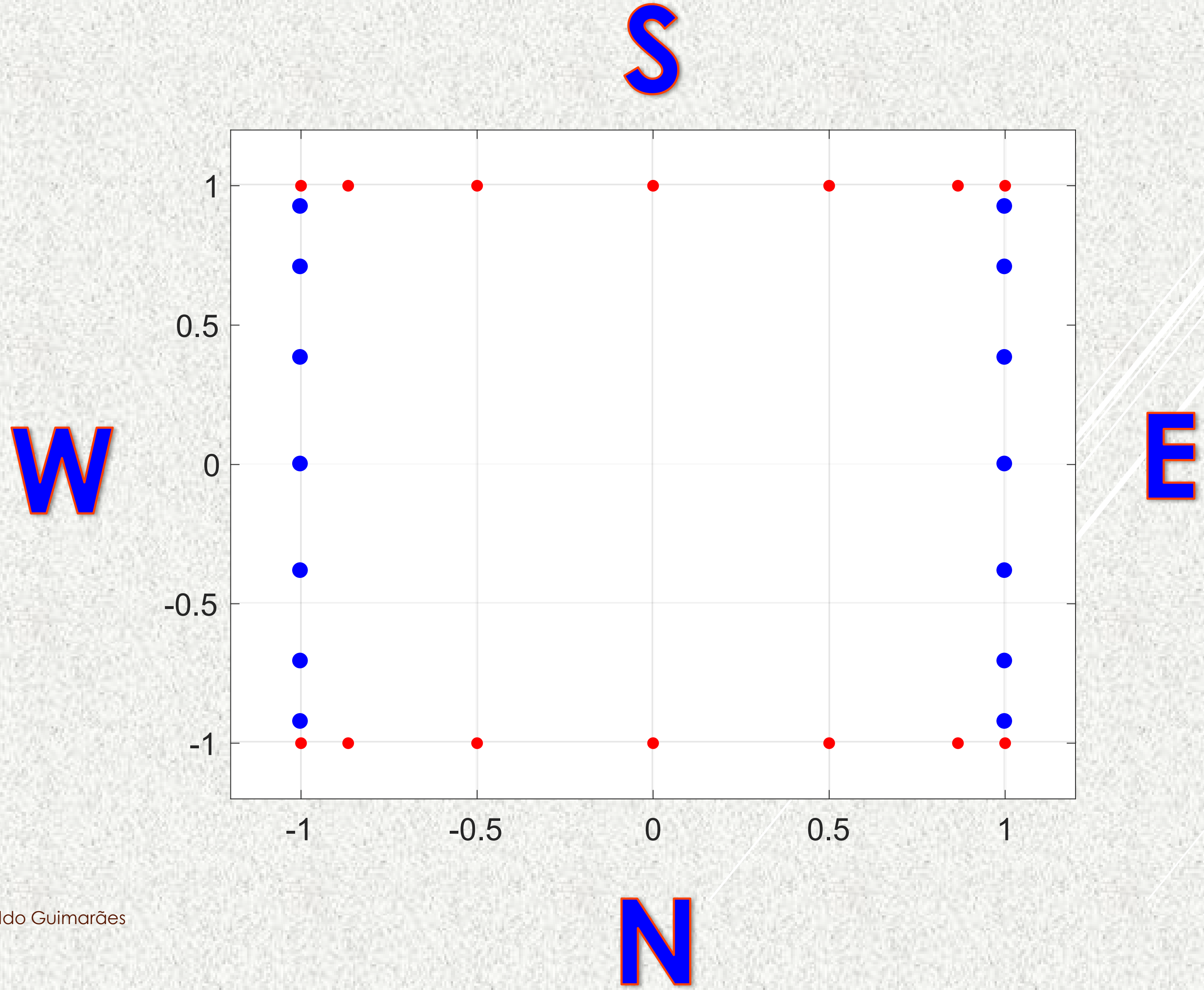
$$D_y^2 F I_x + I_y F\left(D_x^2\right)^T = f$$

$$\text{vec}\left(D_y^2 F I_x\right) + \text{vec}\left(I_y F\left(D_x^2\right)^T\right) = \text{vec}\left(f\right)$$

$$\left[I_x \otimes D_y^2\right] \cdot \text{vec}\left(F\right) + \left[D_x^2 \otimes I_y\right] \cdot \text{vec}\left(F\right) = \text{vec}\left(f\right)$$

$$\underbrace{\left[I_x \otimes D_y^2 + D_x^2 \otimes I_y\right]}_{S} \cdot \text{vec}\left(F\right) = \text{vec}\left(f\right) \rightarrow \text{ Sistemar linear}$$

Contorno: cuidado!



Pontos do grid

$n_y + 1$ para y e $n_x + 1$ para x

Total: $\left(n_y + 1\right) \cdot \left(n_x + 1\right) = n_x n_y + n_x + n_y + 1$

$$\text{Se } n_x = n_y = n \Rightarrow n_{\text{total}} = n^2 + 2n + 1$$

Contorno: $2\left(n_x + 1\right) + 2\left(n_y - 1\right) = 2\left(n_x + n_y\right)$

$$\text{Se } n_x = n_y = n \Rightarrow n_{\text{contorno}} = 4n$$

$$\left(n+1\right)^2 - 4n = \left(n-1\right)^2$$

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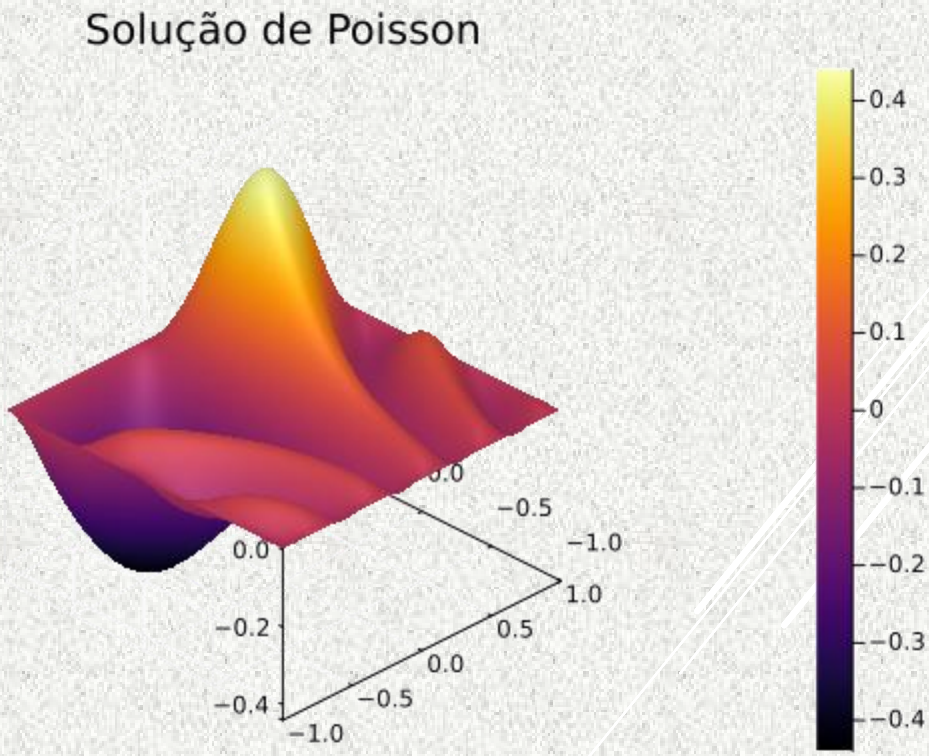
$$\Delta F = 10 \cdot \sin\left[8x\left(y-1\right)\right] \qquad \Omega = \left(x, y\right) \in \left[-1,1\right]^2 \subset \mathbb{R}^2$$

$$F = 0 \text{ in } \partial\Omega$$



Seja o PDE: $\Delta F + F = 3e^{x+y}, \quad (x,y) \in \bar{\Omega} = [-1,1]^2 \subset \mathbb{R}^2.$

Solução analítica: $F_{an} = e^{x+y}$.

a) Resolver numericamente o PDE com $M_x = 15$ e $N_y = 20$.

A condição de contorno é $F(\partial\Omega) = F_{an}(\partial\Omega)$.

b) Interpolar para 101 pontos em cada coordenada e exibir o gráfico com a distribuição do erro.