AULA 08
INTRODUÇÃO AOS MÉTODOS ESPECTRAIS
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Polinômios de Legendre

Sturm-Liouville

$$\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]+l(l+1)P_l(x)=0 \quad x\in\left[-1,1\right]$$

Ortogonalidade

$$\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]+l(l+1)P_l(x)=0 \qquad \times P_m$$

$$-\left\{\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]+m(m+1)P_m(x)\right\}=0 \qquad \times P_l$$

$$P_m(x)\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]-P_l(x)\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]\ldots$$

$$\ldots+[l(l+1)-m(m+1)]P_m(x)P_l(x)=0$$

Os dois primeiros termos podem ser escritos como:

$$\frac{d}{dx}\left[\left(1-x^2\right)\left(P_mP_l'-P_lP_m'\right)\right],$$ cuja integral de -1 até 1 é nula.

$$[l(l+1)-m(m+1)]\int_{-1}^{1}P_m(x)P_l(x)dx=0$$

Portanto, se $m \neq l \Rightarrow \int_{-1}^{1}P_m(x)P_l(x)dx=0$

Prova

$$\frac{d}{dx}\left[\left(1-x^2\right)\left(P_m P_l' - P_l P_m'\right)\right] = \frac{d}{dx}\left[\left(1-x^2\right)\left(P_m P_l'\right)\right] - \frac{d}{dx}\left[\left(1-x^2\right)\left(P_l P_m'\right)\right] =$$

$$= \frac{d}{dx}\left[\left(1-x^2\right)\left(P_l'\right)\right]\cdot P_m + P_m'\left[\left(1-x^2\right)\left(P_l'\right)\right] - \frac{d}{dx}\left[\left(1-x^2\right)\left(P_m'\right)\right]\cdot P_l - P_l'\left[\left(1-x^2\right)\left(P_m'\right)\right] =$$

$$= P_m \cdot \frac{d}{dx}\left[\left(1-x^2\right)\left(P_l'\right)\right] - P_l \frac{d}{dx}\left[\left(1-x^2\right)\left(P_m'\right)\right]$$

Recorrência para a derivada

$$\frac{x^2-1}{n}\frac{d}{dx}P_n(x) = xP_n(x) - P_{n-1}(x)$$

TAREFA-códigos

I) Calcule numericamente as integrais seguintes, com erro relativo menor que uma parte em 100 milhões.

a) $I = \int_0^4 te^{2t}dt \simeq 5.21692648\text{e}3$

b) $I = \int_{-1}^0 \frac{\sin x}{x} dx \simeq 0.946083070$

c) $I = \int_0^2 e^{-x^2} dx \simeq 0.88208139$

d) $I = \int_{-1}^1 \frac{e^{x-1}-1}{x-1} dx \simeq 1.31926336$

Relações importantes para construção das matrizes operacionais

$$\boxed{\int_{-1}^{x} P_n(u)\,du = \frac{P_{n+1} - P_n}{2n+1}}$$

$$D_{i+1,j+1}^{\text{Cheby}} = \begin{cases} j \text{ if } i = 0 \text{ and } j \text{ odd} \\ 2j, \text{ if } 0 < i < j, \quad i + j \text{ odd.} \end{cases}$$

$$D_{i+1,j+1}^{\text{Leg}} = 2i + 1, \text{ if } 0 < i < j, \quad i + j \text{ odd.}$$

Fazer a integral numérica de Cheby-Lobatto, sabendo que

os pesos são dados por: $w_k = \dfrac{\pi}{n}, \quad k = 1 : n-1$

e $\quad w_k = \dfrac{\pi}{2n}$ para $k = 0$ ou $k = n$

a) $f_1(x) = 1$

b) $f_2(x) = x^2$

c) $f_3(x) = \left(2x^2 - 1\right)^2$

d) $f_4(x) = \sqrt{1 - x^2}$

Derivatives of Chebyshev polynomials

$$T'_{2n}(x) = (2n)\cdot 2\sum_{k=1}^{n} T_{2k-1}(x)$$

$$T'_{2n+1}(x) = (2n+1)\left[T_0(x) + 2\sum_{k=1}^{n} T_{2k}(x)\right]$$

Se $m = 2n$, considere a identidade:

$$\frac{\sin(2n\theta)}{\sin\theta} = 2\sum_{k=1}^{n}\cos(2k-1)\theta \qquad \text{verificada por}$$

$$2\sin\theta\cdot\sum_{k=1}^{n}\cos(2k-1)\theta = \sum_{k=1}^{n}2\cos(2k-1)\theta\cdot\sin\theta,$$

como $\left(\cos a\cdot\cos b = \left[\sin(a+b)-\sin(a-b)\right]/2\right)$, segue-se

$$\sin(2n\theta) = \sum_{k=1}^{n}\left[\sin(2k\theta)-\sin(2k-2)\theta\right] = \sum_{k=1}^{n}\left[\sin(2k\theta)-\sin(2(k-1)\theta)\right]$$

$$\sin(2n\theta) = \sum_{k=1}^{n}\sin(2k\theta) - \sum_{\substack{j=1\\ j=k-1}}^{n-1}\sin(2j\theta) = \sin(2n\theta)$$

Se $m = 2n$, considere a identidade:

$$\frac{\sin\left(2n\theta\right)}{\sin\theta} = 2\sum_{k=1}^{n}\cos\left(2k-1\right)\theta \qquad \text{verificada por}$$

$$2\sin\theta\cdot\sum_{k=1}^{n}\cos\left(2k-1\right)\theta = \sum_{k=1}^{n}2\cos\left(2k-1\right)\theta\cdot\sin\theta,\ \text{assim}$$

$$\sin\left(2n\theta\right) = \sum_{k=1}^{n}\left[\sin\left(2k\theta\right)-\sin\left(2k-2\right)\theta\right] = \sum_{k=1}^{n}\left[\sin\left(2k\theta\right)-\sin\left(2(k-1)\theta\right)\right]$$

$$\sin\left(2n\theta\right) = \sum_{k=1}^{n}\sin\left(2k\theta\right)-\sum_{\substack{j=1\\ j=k-1}}^{n-1}\sin\left(2j\theta\right) = \sin\left(2n\theta\right)$$

Se $m = 2n+1$, considere a identidade:

$$\frac{\sin\left(2n+1\right)\theta}{\sin\theta} = 1 + 2\sum_{k=1}^{n}\cos 2k\theta \qquad \text{verificada por}$$

$$\sin\theta + 2\sum_{k=1}^{n}\left[\sin\theta\cdot\cos\left(2k\theta\right)\right] = \sin\theta + \sum_{k=1}^{n}\sin\left(2k+1\right)\theta - \sin\left(2k-1\right)\theta, \text{ assim}$$

$$\sin\left(2n+1\right)\theta = \sum_{k=1}^{n}\sin\left(2k+1\right)\theta - \sum_{\substack{j=1\\ j=k-1}}^{n-1}\sin\left(2j+1\right)\theta = \sin\left(2n+1\right)\theta$$

$$D_{\text{Cheby}} = \begin{bmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
function [D xs] = D_Cheb(n)
%% Matrix of differentiation
D=zeros(n+1);
for i=0:n-1
    for j = i+1:2:n
       D(i+1,j+1)=(2/myC(i))*(j);
    end
end
%% Nodes
xs = -cos((0:n)*pi/(n));
function y=myC(p)
if p == 0
    y=2;
elseif p < 0
    y=0;
else
    y=1;
end
return
```

Exemplo elementar

$y'(x) = 4x \quad y(1) = 1 \quad x \in [-1,1]$

$D\hat{y} = 4T_1$ e $[1\ 1\ 1\ 1\ 1]\cdot \hat{y} = 1$

$$\begin{bmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \hat{y} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{y} = [0\ 0\ 1\ 0\ 0]^T = T_2$$

$$D_{\text{Leg}} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Derivada de um polinômio de Legendre

$$P'_n = \frac{d}{dx}(P_n) = c_0 P_0 + c_1 P_1 + \ldots + c_{n-1} P_{n-1}$$

$$c_k = \frac{2k+1}{2} \int_{-1}^{1} \underbrace{P_k}_{u} \underbrace{P'_n \, dx}_{dv} =$$

$$c_k = \left\{ \left[P_k P_n \right]_{-1}^{+1} - \underbrace{\int_{-1}^{1} P_n P'_k \, dx}_{\text{ortogonais}} \right\} \frac{2k+1}{2}$$

$$c_k = \frac{2k+1}{2} \left[1 - (-1)^{k+n} \right]$$

$$\boxed{c_k = 2k+1 \quad (k+n) \text{ ímpar}}$$

Integração Espectral

I - Chebyshev

$$\int_{-1}^{x} T_n\,dx = \frac{1}{2}\left[\frac{T_{n+1}(x)}{(n+1)} - \frac{T_{n-1}(x)}{(n-1)}\right]_{-1}^{x} = \frac{1}{2}\left[\frac{T_{n+1}(x)}{(n+1)} - \frac{T_{n-1}(x)}{(n-1)}\right] - \frac{1}{2}\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{(n-1)}\right] \Rightarrow$$

$$\int_{-1}^{x} T_n\,dx = \frac{1}{2}\left[\frac{T_{n+1}(x)}{(n+1)} - \frac{T_{n-1}(x)}{(n-1)}\right] - \frac{(-1)^{n+1}}{2}\left(\frac{n-1-n-1}{n^2-1}\right) = \frac{1}{2}\left[\frac{T_{n+1}(x)}{(n+1)} - \frac{T_{n-1}(x)}{(n-1)}\right] + \frac{(-1)^{n+1}}{n^2-1}$$

II - Legendre

$$\int_{-1}^{x} P_n\,dx = \left[\frac{P_{n+1} - P_{n-1}}{2n+1}\right]_{-1}^{x} = \frac{P_{n+1} - P_{n-1}}{2n+1}$$

III - Fourier

$$\int_{0}^{\theta} e^{ik\theta}\,dt = \left[\frac{e^{ik\theta}}{ik}\right]_{0}^{\theta} = \overbrace{-i\left[\frac{e^{ik\theta}}{k} - \frac{1}{k}\right]}^{k\neq 0} + \underbrace{\theta}_{k=0}$$

Integração Espectral-Matrizes

$$J_{\text{Cheby}} = \begin{bmatrix} 1 & -1/4 & -1/3 & 1/8 & -1/15 \\ 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1/4 & 0 & -1/4 & 0 \\ 0 & 0 & 1/6 & 0 & -1/6 \\ 0 & 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & 0 & 1/10 \end{bmatrix}$$

$$J_{\text{Legendre}} = \begin{bmatrix} 1 & -1/3 & 0 & 0 & 0 \\ 1 & 0 & -1/5 & 0 & 0 \\ 0 & 1/3 & 0 & -1/7 & 0 \\ 0 & 0 & 1/5 & 0 & -1/9 \\ 0 & 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 0 & 1/9 \end{bmatrix}$$

$$J_{\text{Fourier}} = i \cdot \begin{bmatrix} 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/3 & -1/2 & -1 & 0 & 1 & 1/2 & 1/3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/3 \end{bmatrix}$$

Fazer os códigos

Physical Integration

Polynomial Bases

$$J_{\text{Phys}} = \underbrace{B_{n+1}\left(x_0^n\right)}_{n+1,n+2} \cdot \underset{n+2,n+1}{J_S^n} \cdot \underbrace{\left(B_n\left(x_0^n\right)\right)}_{n+1,n+1}^{-1}$$

Pesos da integração numérica definida

$$w_k = J^{\text{Phys}}_{(n+1,k)}$$

Fourier $t \in [0, 2\pi] \subset \mathbb{R}$

$$J_{\text{Phys}} = B_M \cdot J_S \cdot \left(B_M\right)^{-1} + A_0 \cdot t$$

Quadraturas

- Fourier – Trapezoidal
- Clenshaw-Curtis
- Frejét
- Lobatto
- Radau
- Gauss
- Fourier

C. W. Clenshaw and A. R. Curtis, *A method for numerical integration on an automatic computer*, Numer. Math., 2 (1960), pp. 197–205.

Pesos p/ interpolação baricêntrica

Chebyshev-Lobatto

$$x_k = -\cos(k\pi / n), \quad k = 0,1,\ldots,n \quad \text{então:}$$

$$\begin{cases} w_k = \left[\dfrac{1}{2}, -1, 1, -1, \ldots, -\dfrac{1}{2}\right] & n \text{ ímpar} \\ w_k = \left[-\dfrac{1}{2}, 1, -1, 1, \ldots, -\dfrac{1}{2}\right] & n \text{ par} \end{cases}$$

Fourier com N pontos em $[0,2\pi)$

$$x_k = \frac{2\pi}{N}k, \quad k = 0,1,\ldots,N-1.$$

$$w_k = (-1)^k$$

Autovalores e autovetores

Equação com $x \in [-1,1] \subset \mathbb{R}$

$y'' + k^2 y = 0, \qquad y(\pm 1) = 0$

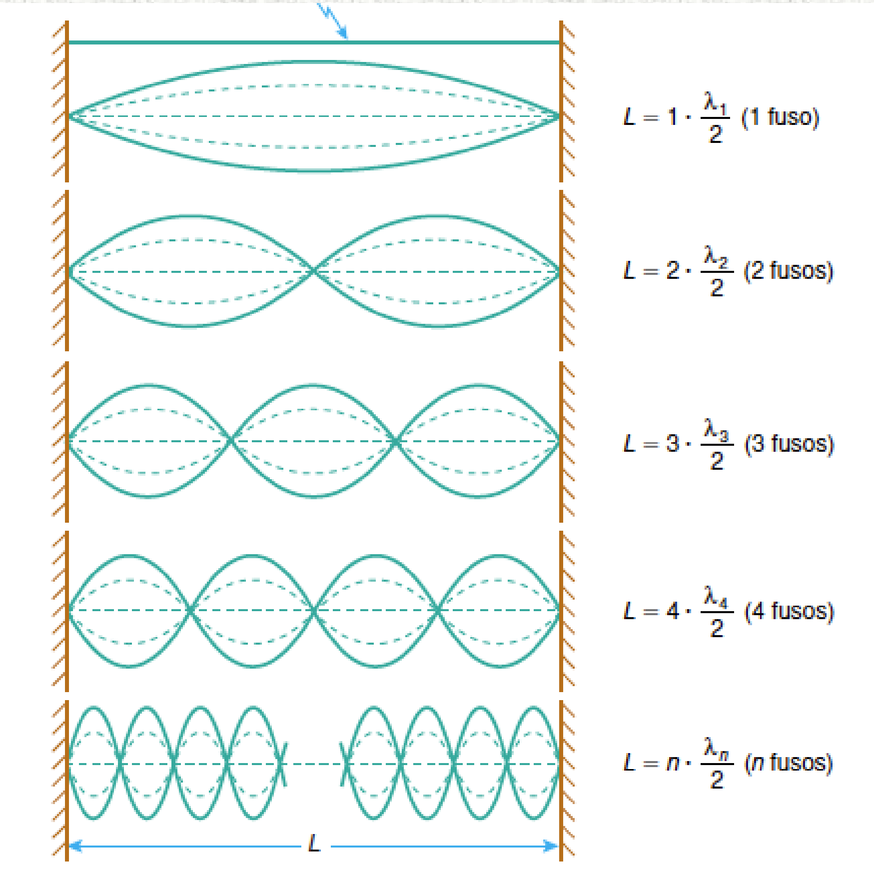
$y'' = -k^2 y$

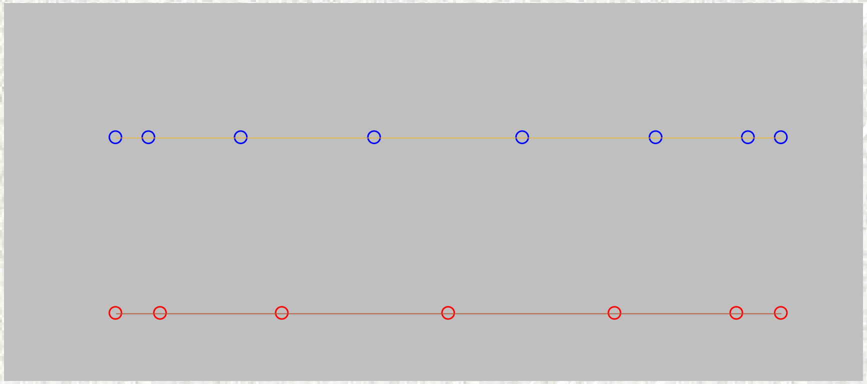
Tem solução trivial!

```
% p15.m - solve eigenvalue BVP u_xx = lambda*u, u(-1)=u(1)=0

  N = 36; [D,x] = cheb(N); D2 = D^2; D2 = D2(2:N,2:N);
  [V,Lam] = eig(D2); lam = diag(Lam);
  [foo,ii] = sort(-lam);              % sort eigenvalues and -vectors
  lam = lam(ii); V = V(:,ii); clf
  for j = 5:5:30                      % plot 6 eigenvectors
    u = [0;V(:,j);0]; subplot(7,1,j/5)
    plot(x,u,'.','markersize',12), grid on
    xx = -1:.01:1; %uu = polyval(polyfit(x,u,N),xx);
    uu = Bary_Cheby(u,-xx);
    line(xx,uu), axis off
    text(-.4,.5,sprintf('eig %d =%20.13f*pi^2/4',j,lam(j)*pi^2/4))
    text(.7,.5,sprintf('%4.1f  ppw', 4*N/(pi*j)))
  end
```

Pontos por comprimento de onda





$$\begin{cases} n \text{ par} & \Delta x_{\text{máx}} = \sin\left(\frac{\pi}{n}\right) \simeq \frac{\pi}{n} \\ n \text{ ímpar} & \Delta x_{\text{máx}} = 2\sin\left(\frac{\pi}{2n}\right) \simeq \frac{\pi}{n} \end{cases}$$

? Demonstrar

Assim, $\frac{\lambda}{\Delta x_{\text{máx}}} = \frac{4}{j} \cdot \frac{n}{\pi} \Rightarrow \qquad ppw = \frac{4n}{\pi j}$

Temos $\lambda_j = \frac{2L}{j}$, como $L = 2$: $\lambda_j = \frac{4}{j}$