AULA 07
INTRODUÇÃO AOS MÉTODOS ESPECTRAIS
PTC 5525 (30/10/2025)

Osvaldo Guimarães

Polinômios de Legendre

Sturm-Liouville

$$\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]+l(l+1)P_l(x)=0 \quad x\in\left[-1,1\right]$$

Ortogonalidade

$$\frac{d}{dx}\left[\left(1-x^2\right)P_\ell'(x)\right]+\ell(\ell+1)P_\ell(x)=0 \qquad \times P_m$$

$$-\quad\left\{\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]+m(m+1)P_m(x)\right\}=0 \qquad \times P_\ell$$

$$P_m(x)\frac{d}{dx}\left[\left(1-x^2\right)P_\ell'(x)\right]-P_\ell(x)\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]\ldots$$

$$\ldots+[\ell(\ell+1)-m(m+1)]P_m(x)P_\ell(x)=0$$

Os dois primeiros termos podem ser escritos como:

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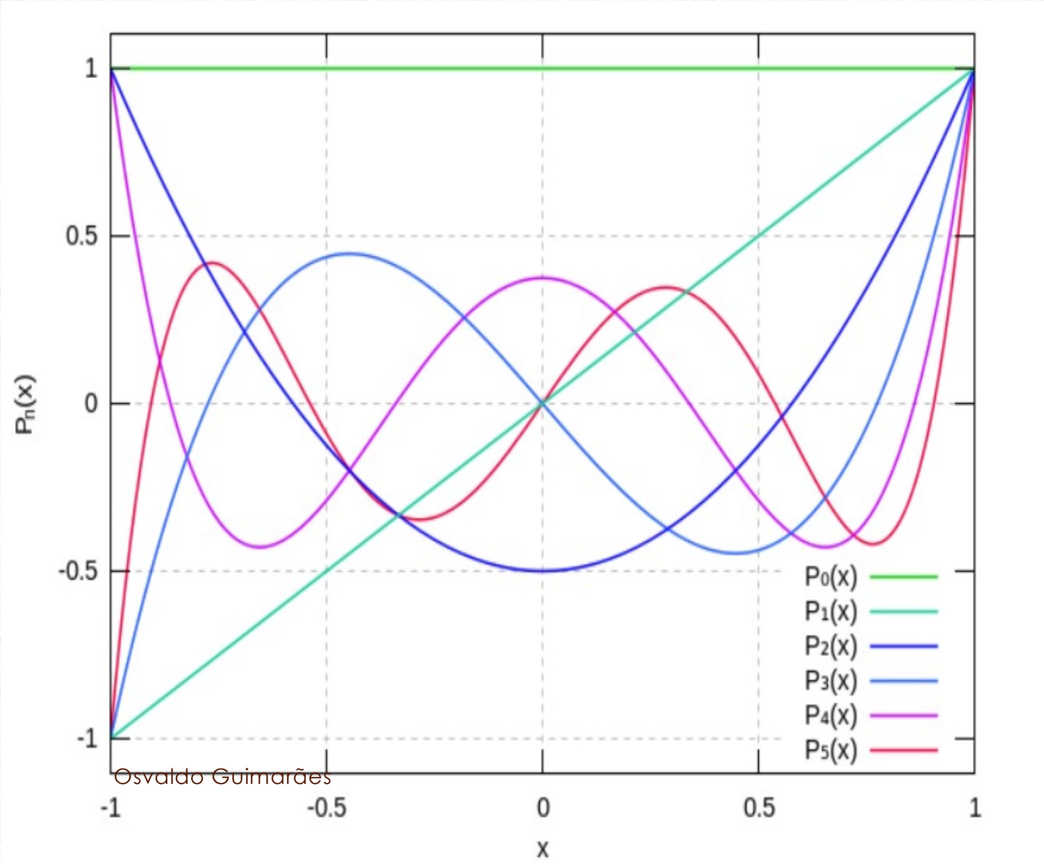
$\frac{d}{dx}\left[\left(1-x^2\right)\left(P_mP_\ell'-P_\ell P_m'\right)\right]$, cuja integral de -1 até 1 é nula.

$$[\ell(\ell+1)-m(m+1)]\int_{-1}^{1}P_m(x)P_\ell(x)dx=0$$

Portanto, se $m\neq\ell\Rightarrow \qquad \int_{-1}^{1}P_m(x)P_\ell(x)dx=0$

Polinômios de Legendre

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1}\delta_{mn} \qquad x \in [-1,1] \subset {}^{\sim}$$



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n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

$$P_0 = 1,\quad P_1 = x,\quad P_n(1) = 1 \qquad P_n(-1) = (-1)^n \qquad \int_{-1}^{1} P_k(x)\cdot dx = 0\,(k \neq 0)$$

Recorrência: $(n+1)P_{n+1} = x(2n+1)P_n - nP_{n-1}$

$$P_n = \frac{P'_{n+1} - P'_n}{2n+1} \Rightarrow \int_{-1}^{x} P_n(u)\,du = \left.\frac{P_{n+1} - P_n}{2n+1}\right|_{-1}^{x}$$

Legendre como base completa no espaço de Hilbert

$$\lim_{n\to\infty}\int_a^b \left(f_{\text{Analytical}}(x) - \sum_{k=0}^{n} c_k \cdot P_k(x) \right)^2 \cdot dx = 0$$

$$f_n = \left\langle B_n \middle| c_n \right\rangle \qquad \text{Cada } c_k \text{ é obtido por: } c_k = \frac{2k+1}{2}\int_{-1}^{1} f \cdot P_k dx.$$

$$\text{Isto é, } c_k = \left\langle P_k \middle| f \right\rangle \cdot \frac{2k+1}{2} \text{ (projetor): } \quad \frac{?\left|P_k\right\rangle}{\left\langle P_k \middle| P_k \right\rangle}.$$

Interesse meramente teórico

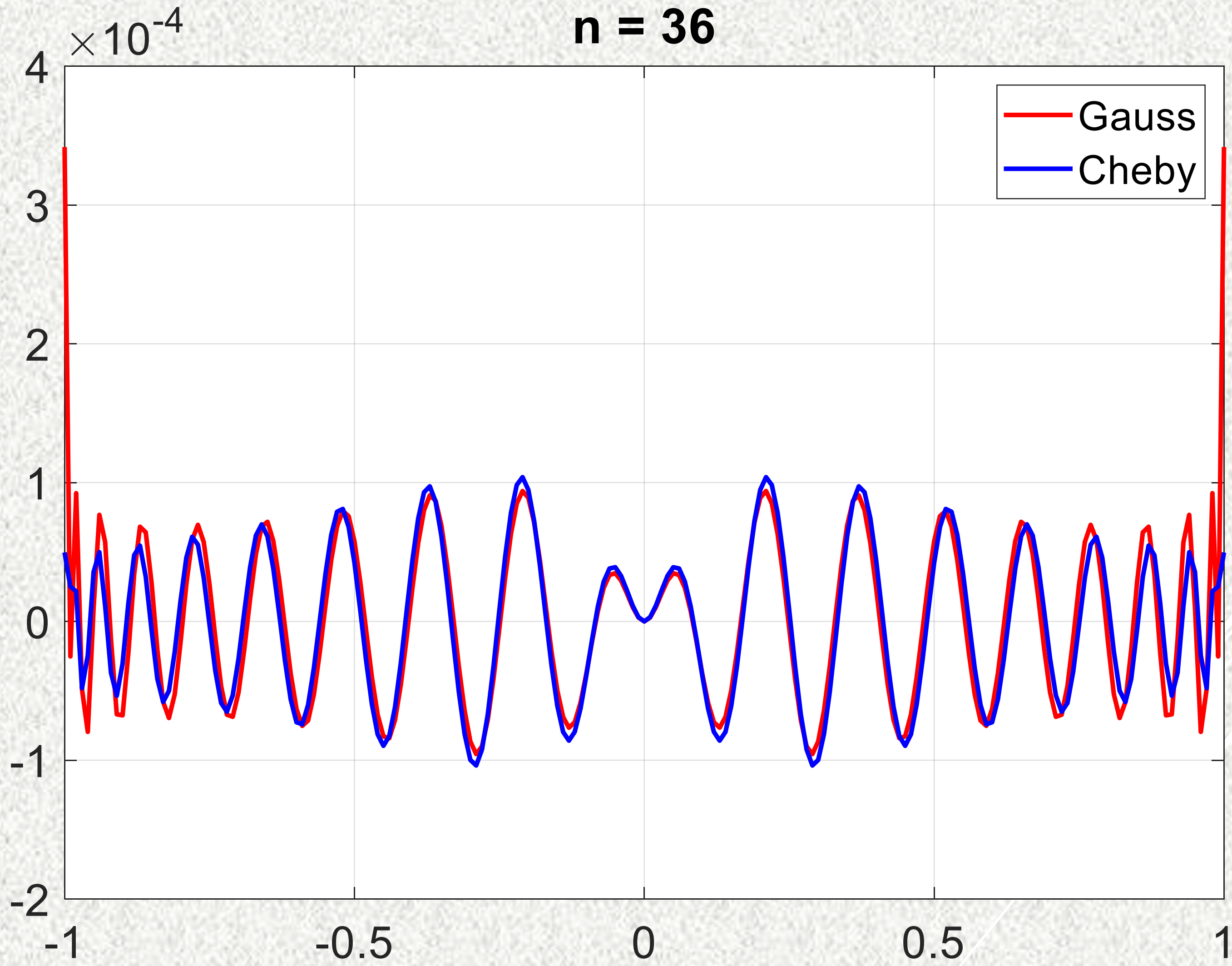
$$x^n = \sum_{k=n,n-2,\dots} \frac{(2k+1)n!}{2^{(n-k)/2}\left(\frac{1}{2}(n-k)\right)!(k+n+1)!!} P_k(x)$$

Qualquer P_k de Legendre é uma combinação linear de monômios.

Reciprocamente, qualquer polinômio de grau n é uma combinação linear de de $P_{k=0:n}$.

Interpolação de: $f(x)=\frac{1}{1+16x^2}$

$$I_n=\sum w_i \cdot f(x_i),\quad i=0:n$$



Cheby - Type I nodes

$$\theta_k=\frac{(2k+1)\pi}{2(n+1)},\ k=0:n$$

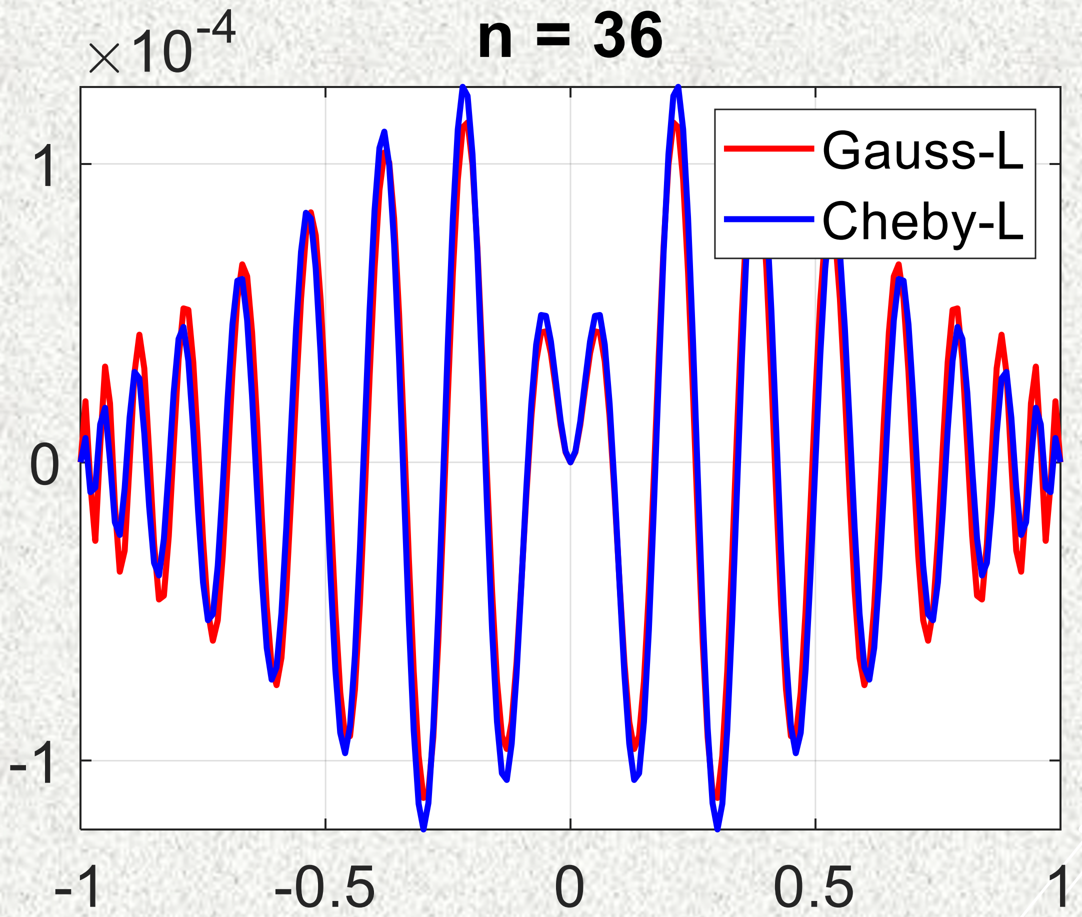
$$x_k=\cos(\theta_k)$$

Legendre -

Raizes de P_{n+1}

Interpolação de: $f(x)=\dfrac{1}{1+16x^2}$

$$I_n=\sum w_i\cdot f(x_i),\quad i=0:n$$



Lobatto points

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Quadratura de Gauss-Legendre

$$I_n = \int_{-1}^{1} f\,dx = \sum w_i \cdot f\left(x_i\right),\ \ i = 1 : n+1,$$

c/ $2n+2$ incógnitas.

Consideremos que f seja polinomial de grau $2n + 1$.

Podemos efetuar a divisão de polinômios: $\dfrac{P_{v(2n+1)}}{Q_{n+1}} \quad \Rightarrow P_v = P_n \cdot Q_{n+1} + R_n.$

$$\int P_v = \underbrace{\int P_n \cdot Q_{n+1}}_{=0} + \int R_n$$

Q_{n+1} é de Legendre

Como os pesos não dependem de R_n, temos $\int R_n = w_k R_n\left(x_k\right)$.

Neste ponto, substituímos R_n e: $\int P_v = w_k\left[P_v\left(x_k\right) - P_n\left(x_k\right) \cdot Q_{n+1}\left(x_k\right)\right]$.

Se escolhemos os x_k de forma que sejam as raízes de Q_{n+1}, então $\int P_v = w_k \cdot P_v\left(x_k\right)$, exatamente.

Quem são as raízes de um Q_{n+1}?

$$(n+1)P_{n+1} = x(2n+1)P_n - nP_{n-1}$$

$$xP_k = \frac{k}{2k+1}P_{k-1} + \frac{k+1}{2k+1}P_{k+1}$$

$$x\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} = \begin{pmatrix} 0 & 1 & & & 0 \\ 1/3 & 0 & 2/3 & & \\ & 2/5 & 0 & 3/5 & \\ \vdots & & \ddots & & \vdots \\ 0 & & & \frac{n-1}{2n-1} & 0 \end{pmatrix} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} + \frac{n+1}{2n+1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ P_{n+1} \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ 1/3 & 0 & 2/3 & & \\ & 2/5 & 0 & 3/5 & \\ & & & & \\ 0 & & & \frac{n-1}{2n-1} & 0 \end{pmatrix}}_{M} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} = x_j \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} - \frac{n+1}{2n+1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ P_{n+1}(x_j) \end{bmatrix}$$

Portanto, $M \cdot \vec{p}_{n-1} = \lambda \cdot \vec{p}_{n-1} + C \cdot P_{n+1}$.

Quando é que $M \cdot \vec{p}_{n-1} = x_j \cdot \vec{p}_{n-1}$, exatamente?

Se, e somente se, x_j for uma das raízes de $P_{n+1}(x)$.

Determinando os n autovalores de M, determinamos as raízes de $P_{n+1}(x)$.

$$w_i = \frac{2}{(1-x_i^2)\,[P'_n(x_i)]^2}.$$

$$
xP_k = \frac{k}{2k+1} P_{k-1} + \frac{k+1}{2k+1} P_{k+1} \Rightarrow
$$

$$
(28) \quad x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{N-1}(x) \end{pmatrix} = \begin{pmatrix} \beta_0 & \frac{k_0}{k_1} & 0 & 0 & \cdots & 0 \\ \frac{k_0}{k_1} & \beta_1 & \frac{k_1}{k_2} & 0 & \cdots & 0 \\ 0 & \frac{k_1}{k_2} & \beta_2 & \frac{k_2}{k_3} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \beta_{N-1} \end{pmatrix} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{N-1}(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{k_{N-1}}{k_N}\phi_N(x) \end{pmatrix}
$$

Pesos:
Identidade de Christoffel–Darboux

$$
w_k = 2V_{1,k}^2
$$

H. S. Wilf, *Mathematics for the Physical Sciences*. New York, NY: Dover Publications, 1962.

J. H. Golub, G. H.; Welsch, "Calculation of Gauss quadrature rules," *Math. Comput.*, 1969.

gauss.m

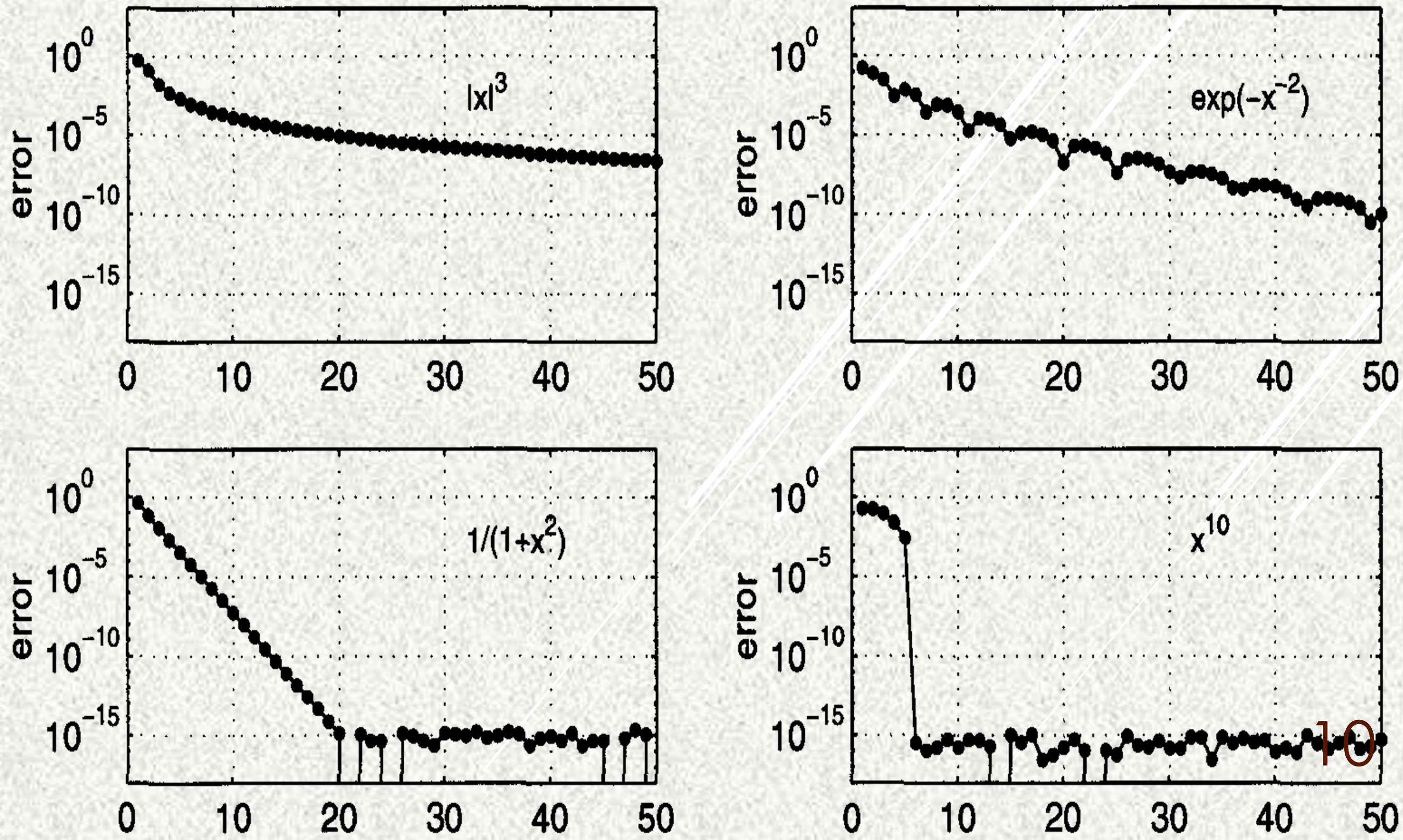
```
% GAUSS  nodes x (Legendre points) and weights w
%        for Gauss quadrature

  function [x,w] = gauss(N)
  beta = .5./sqrt(1-(2*(1:N-1)).^(-2));
  T = diag(beta,1) + diag(beta,-1);
  [V,D] = eig(T);
  x = diag(D); [x,i] = sort(x);
  w = 2*V(1,i).^2;
```

Output 30c

Gauss: $x \in (-1;1)$

Lobatto: $x \in [-1;1]$

Radau: $x \in (-1;1]$ ou $x \in [-1;1)$



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Quadratura de Gauss-Legendre-Radau

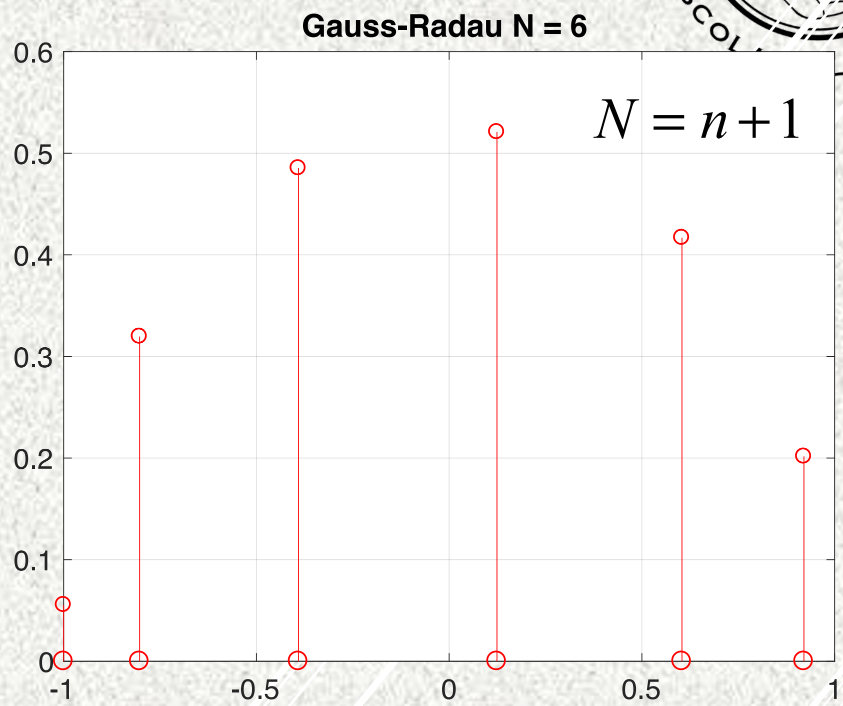
$$I_n = \int_{-1}^{1} f dx = \sum w_i \cdot f(x_i), \quad i = 0:n,$$

c/ $2n+1$ incógnitas, pois $x_0 = -1$: $x_{\text{free}} \in [-1,1)$

x_{free} são as raízes de $\dfrac{P_n + P_{n+1}}{1+x}$.

Os pesos w_i são:

$$w_i = \frac{1-x_i}{(n+1)^2 \left[P_n(x_i)\right]^2} \Rightarrow \quad w_0 = \frac{2}{(n+1)^2}$$



Quadratura de Gauss-Legendre-Radau flipped

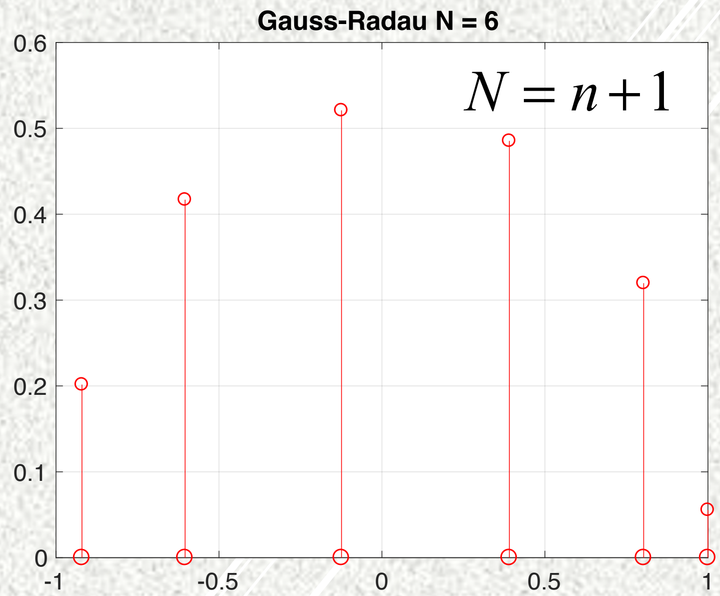
$$I_n = \int_{-1}^{1} f dx = \sum w_i \cdot f\left(x_i\right), \;\; i = 0:n,$$

c/ $2n+1$ incógnitas, pois $x_n = 1$: $x_{\text{free}} \in (-1,1]$

Sendo u_i as raízes de $\dfrac{P_n + P_{n+1}}{1+x}$, $x_i = -u_{n-i}$

Os pesos w_i são os do caso anterior "flipados":

$$w_i = w_{n-i}^{\text{old}}.$$



Quadratura de Gauss-Legendre-Lobatto

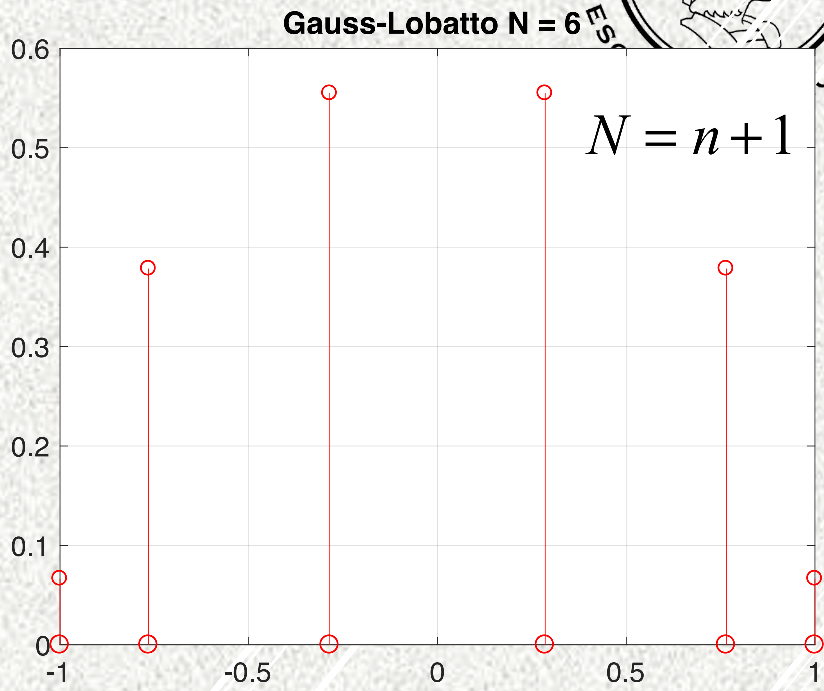
$$I_n = \int_{-1}^{1} f dx = \sum w_i \cdot f\left(x_i\right), \quad i = 0:n,$$

c/ $2n$ incógnitas, pois $x_0 = -1$ e $x_n = 1$: $x_{\text{free}} \in (-1,1)$

x_{free} são as raízes de $P_n'\left(x\right)$.

Os pesos w_i são:

$$w_i = \frac{2}{n\left(n+1\right) P_n^2\left(x\right)}$$



Quadratura de Gauss-Lobatto p/ P_n, dados N pontos $\left(N = n+1\right)$

$$\boxed{w_i = \frac{2}{n\left(n+1\right)\left[P_n\left(x_i\right)\right]^2}}$$

$$\left\langle P_n \middle| P_n \right\rangle_n = \sum w_k \cdot P_n^2\left(x_k\right) = \sum_{k=0}^{n} \frac{2\cancel{P_n^2\left(x_k\right)}}{n\left(n+1\right)\cancel{P_n^2\left(x_k\right)}} = \frac{2}{n}$$

Como $\left\langle P_k \middle| P_k \right\rangle_\infty = \dfrac{2}{2k+1}$, o resultado acima deveria ser:

$$\left\langle P_n \middle| P_n \right\rangle = \frac{2}{2n+1}, \quad \text{logo} \quad \int_{-1}^{1} P_n^2\left(x\right) dx = \frac{n}{2n+1} \sum w_k \cdot P_n^2\left(\left\{x_k\right\}_0^n\right)$$

```
% Find roots of Pn = C_0 + C1*x + C2*x^ +Cn*x^n
%% vv are the coeffs of Pn in crescent order
function rr = Poly_Roots(vv)

%%
vv = vv(:).'; n = length(vv); vv = vv/vv(n);

M = diag(ones(1,n-2),1);
M(n-1,:) = -vv(1:n-1);

rr = sort(eig(M,'vector'));
```

Pesquisar:

$$\theta_k = \frac{(2k+1)\pi}{2N}$$

Cheby-Gauss: $x_k = \cos(\theta_k)$

Chebyshev-Gauss

Chebyshev-Radau

TAREFA-códigos

I) Calcule numericamente as integrais seguintes, com erro relativo menor que uma parte em 100 milhões.

a) $I = \int_0^4 te^{2t} dt$

b) $I = \int_{-1}^0 \frac{\sin x}{x} dx$

c) $I = \int_0^2 e^{-x^2} dx$

d) $I = \int_{-1}^1 \frac{e^{x-1} - 1}{x-1} dx$

TAREFA-códigos

II)

1) Obter a matriz $\left(B_L\right)_{n+1,n+1}$ que aplicada aos $n+1$ coeficientes de uma expansão de Legendre, nos fornece os valores da série nos pontos de Gauss-Lobatto.

2) Obter a matriz $\left(B_L\right)_{n+1,n+1}^{-1}$ que aplicada ao vetor de valores de uma série de Legendre nos pontos de Gauss-Lobatto, nos dá os coeficientes da série.

3) Verificar que $\left(B_L\right)\cdot\left(B_L\right)^{-1}=\left(B_L\right)^{-1}\cdot\left(B_L\right)=I$