AULA 06
INTRODUÇÃO AOS MÉTODOS ESPECTRAIS
PTC 5525 (23/10/2025)

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Resolva pelo método de Newton

$u'' = e^u \qquad u(\pm 1) = 1 \quad x \in [-1,1]$

Considerando $\mathcal{R}(x) = y'' - e^y$, faça o gráfico $R \times x$.

Verifique a solução geral: $u = \log\left(\dfrac{a^2}{1+\cos(ax)}\right)$

S : system

$$S_m = \underline{\bar{I}}\left[D^2 u - e^u\right] \qquad r_m = 0$$

$$J_m = \frac{\partial S_m}{\partial u} = \underline{\bar{I}}\left[D^2 - \operatorname{diag}\left(e^u\right)\right]$$

$$BC\begin{cases} u(-1) = 1 & J(n,1) = \dfrac{\partial u_0}{\partial u_0} = 1 \\ u(+1) = 1 & J(n+1,n+1) = \dfrac{\partial u_n}{\partial u_n} = 1 \end{cases}$$

Método das linhas (MOL)

$$u_t = C_d \cdot \partial_{xx} u$$

$$u_{(0,x)} = \sin\left(\pi x/2\right),\ x \in \left[0,1\right]$$

$$u(t,0) = 0 \qquad u_x\left(t,1\right) = 0$$

Solução analítica: $u = C_d \cdot e^{-\frac{\pi^2}{4}t} \cdot \sin\left(\pi x/2\right)$

Forward Euler: (explicit, first order accurate)

$$u_{n+1} = u_n + \Delta t\, f(u_n)\,.$$

Runge–Kutta-4 (RK4): (explicit, fourth order accurate)

$$\begin{aligned} k_1 &= \Delta t\, f(u_n)\,, \\ k_2 &= \Delta t\, f\Big(u_n + \frac{1}{2}\, k_1\Big)\,, \\ k_3 &= \Delta t\, f\Big(u_n + \frac{1}{2}\, k_2\Big)\,, \\ k_4 &= \Delta t\, f(u_n + k_3)\,, \\ u_{n+1} &= u_n + \frac{1}{6}\Big[k_1 + 2\,k_2 + 2\,k_3 + k_4\Big]\,. \end{aligned}$$

Equação da difusão

$$\frac{\partial U}{\partial t} = C_d \cdot \frac{\partial^2 U}{\partial x^2}$$

```
% Script to solve time dependent PDE
% Cd is the diffusion coefficient
function PDE_meuCd(Cd)

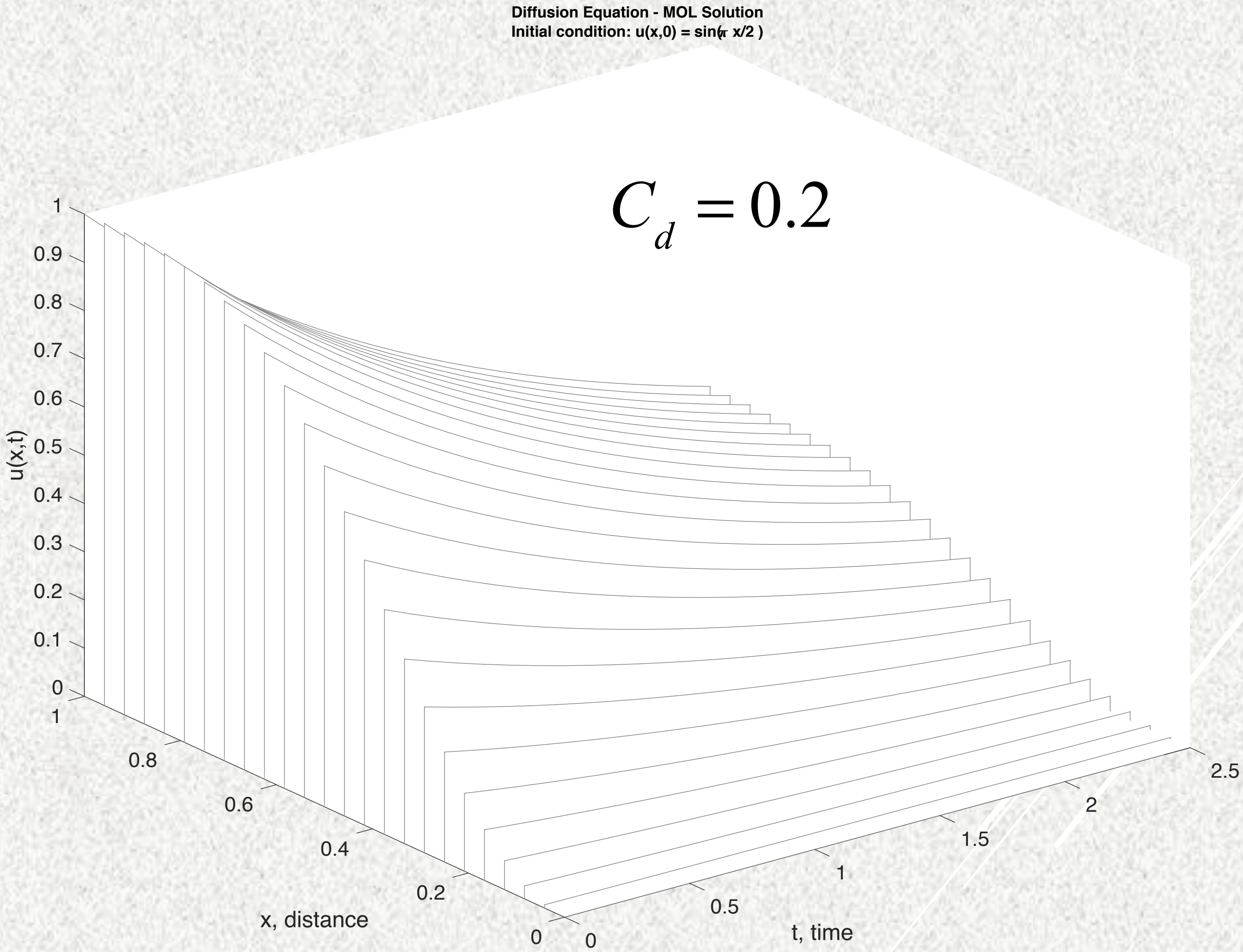
%% Typical Cd = 0.2 - Observe the behavior with Cd = 1.0
tic
N = 25; n = N-1;% N odd - With N = 21 ~ 100
xL = -cos((0:n)'*pi/n); xs = (xL+1)/2;
u0 = sin(pi*xs/2);
%%
D1 = Generalized_Diff_Mat(xs); D1n = D1; D1n(N,:) = 0;
D2 = D1*D1n;
uprime = @(t,u) Cd*D2*[0;u(2:end)];
%% Time vector
Nt = N;
t0=0.0;   tf=2.5; tout=linspace(t0,tf,Nt); nout = Nt;
tic
reltol=1.0e-11; abstol=1.0e-11;
  options=odeset('RelTol',reltol,'AbsTol',abstol);
  [t,u]=ode45(uprime,tout,u0,options); %Runge-Kutta
toc
```

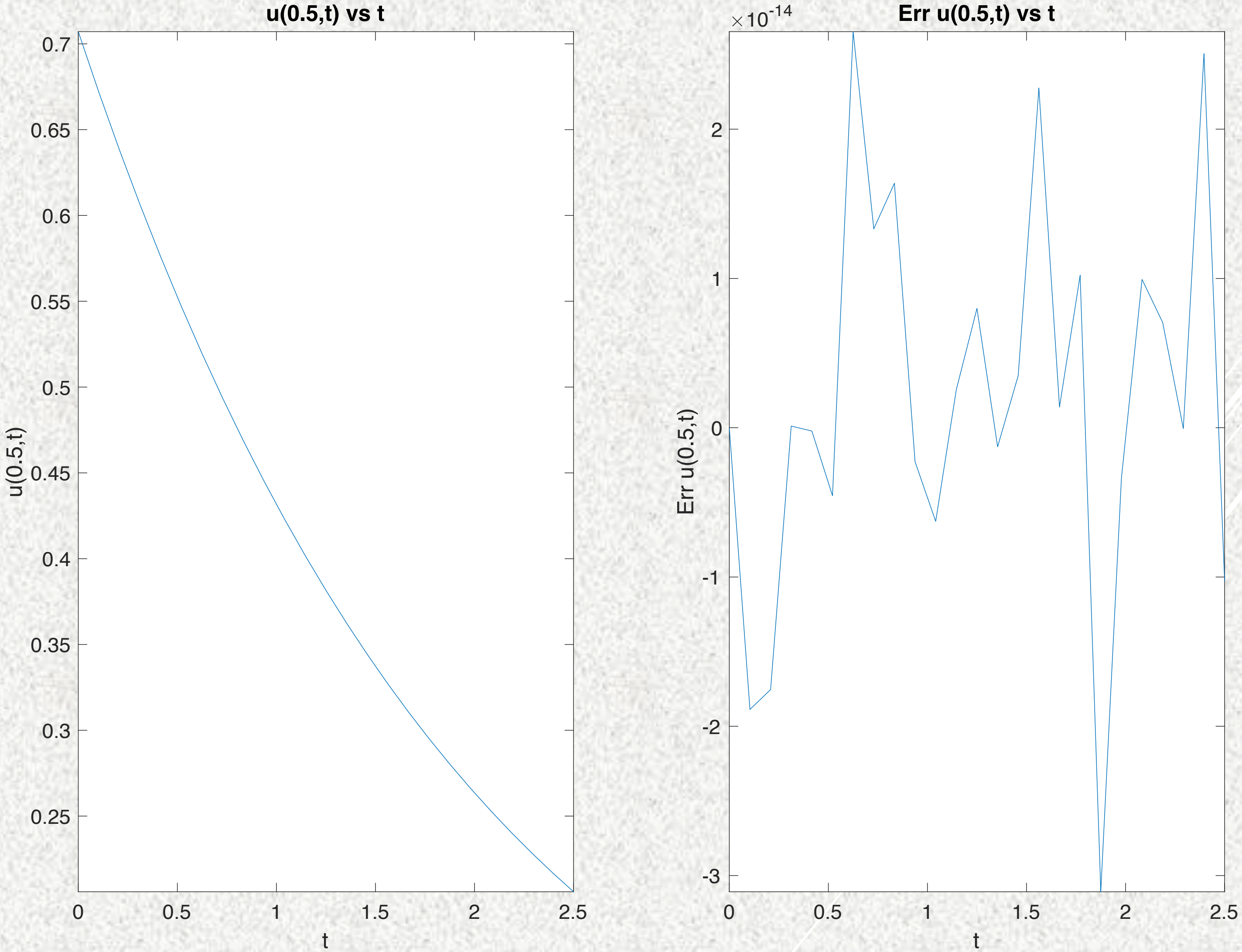
A página seguinte descreve somente os plots.

```
%%
[xx,tt] = meshgrid(xs,tout);
u_Exact = @(t,x) exp(-Cd*pi^2/4.0*t).*sin(pi*x/2);
figure (3);
surf(tt,xx,u_Exact(tt,xx) - u);
xlabel t; ylabel x
%%
n2=n/2+1;   sine=sin(pi/2.0*0.5);
for i=1:nout
  u_plot(i)=u(i,n2);
  u_anal(i)=exp(-Cd*pi^2/4.0*t(i))*sine;
  err_plot(i)=u_plot(i)-u_anal(i);
end
% Display selected output
fprintf('\n  abstol = %8.1e   reltol = %8.1e\n',...
        abstol,reltol);
fprintf('\n     t        u(0.5,t)  u_anal(0.5,t)   err u(0.5,t)\n');
for i=1:5:nout
  fprintf('%6.3f%15.6f%15.6f%15.7f\n',...
          t(i),u_plot(i),u_anal(i),err_plot(i));
end
```

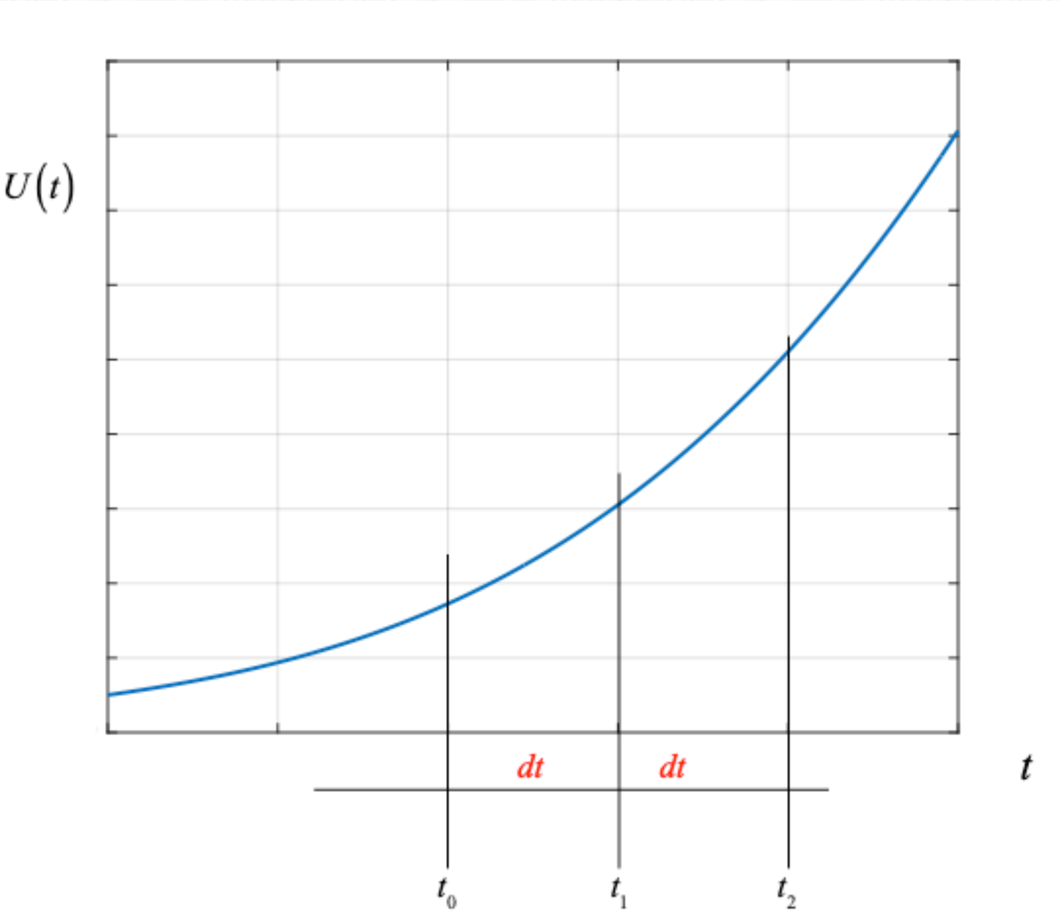
```
toc
% Plot numerical solution and errors at x = 1/2
  figure (1);
  subplot(1,2,1)
  plot(t,u_plot); axis tight
  title('u(0.5,t) vs t'); xlabel('t'); ylabel('u(0.5,t)')

  subplot(1,2,2)
  plot(t,err_plot); axis tight
  title('Err u(0.5,t) vs t'); xlabel('t'); ylabel('Err u(0.5,t)')
%% Plot numerical solution in 3D perspective
  figure(2);
  colormap('Gray');
  C=ones(N,Nt);
  g=linspace(0,1,N); % For distance x
  waterfall(t,g,u',C);
  axis('tight');
  grid off
  xlabel('t, time')
  ylabel('x, distance')
  zlabel('u(x,t)')
  s1 = sprintf('Diffusion Equation - MOL Solution');
  sTmp = sprintf('u(x,0) = sin(\\pi x/2 )');
  s2 = sprintf('Initial condition: %s', sTmp);
  title([{s1}, {s2}], 'fontsize', 12);
```


Diffusion Equation - MOL Solution
Initial condition: u(x,0) = sin(π x/2)
$C_d = 0.2$
u(x,t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x, distance
t, time
0.2
0.4
0.6
0.8
0.5
1
1.5
2
2.5


u(0.5,t) vs t
u(0.5,t)
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0
0.5
1
1.5
2
2.5
t
Err u(0.5,t) vs t
$\times 10^{-14}$
Err u(0.5,t)
2
1
0
-1
-2
-3
0
0.5
1
1.5
2
2.5
t

Leap frog



$$\dot{U}_2 = \frac{U_2 - U_1}{dt}, \text{ and } \dot{U}_1 = \frac{U_1 - U_0}{dt}$$

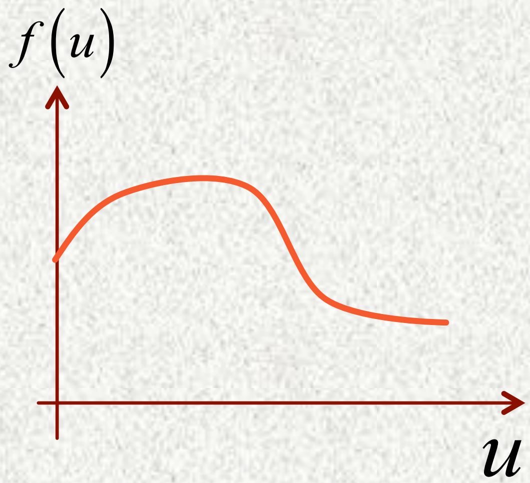
$$\ddot{U}_1 = \frac{\dot{U}_2 - \dot{U}_1}{dt} = \frac{U_2 - U_1 - U_1 + U_0}{dt^2}$$

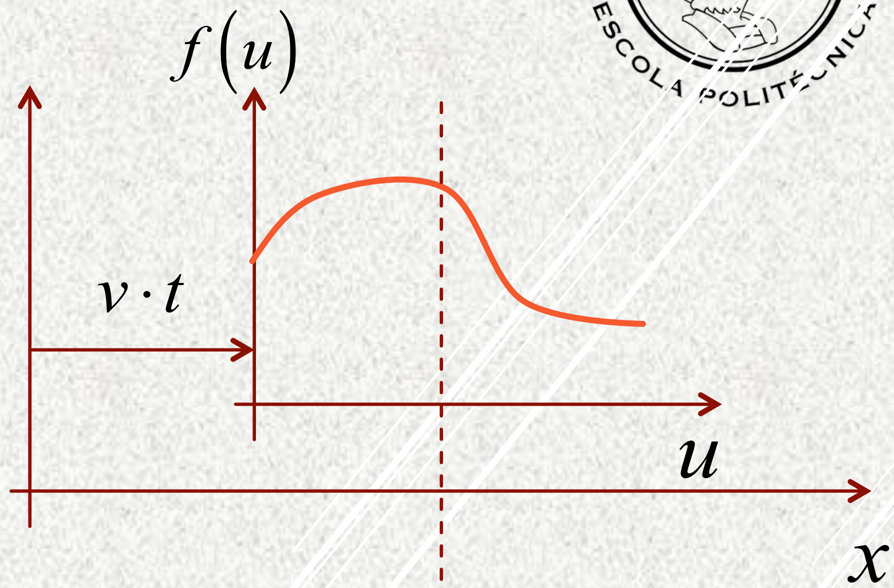
$$\ddot{U}_1 \cdot dt^2 = U_2 - 2U_1 + U_0$$

$$\Rightarrow U_2 = 2U_1 - U_0 + \ddot{U}_1 \cdot dt^2$$

Para a 2ª derivada

Equação da onda





Para cada abscissa u, temos: $f(u) = f(x - vt)$

$x = u + v \cdot t \Rightarrow u = x - vt$

$\frac{\partial u}{\partial x} = 1$ e $\frac{\partial u}{\partial t} = -v$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(-v)$$

$$\boxed{\frac{\partial f}{\partial t} = -v \cdot \frac{\partial f}{\partial x}}$$

Portanto:

$$\frac{\partial f}{\partial t} = -v \cdot \frac{\partial f}{\partial x}$$

Onda regressiva: $f = f(x + v \cdot t)$

Em ambos os casos, f é arbitrária.

Para cada abscissa u, temos: $f(u) = f(x - vt)$

$$x = u + v \cdot t \Rightarrow u = x - vt$$

$$\frac{\partial u}{\partial x} = 1 \quad \text{e} \quad \frac{\partial u}{\partial t} = -v$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial^2 u}$$

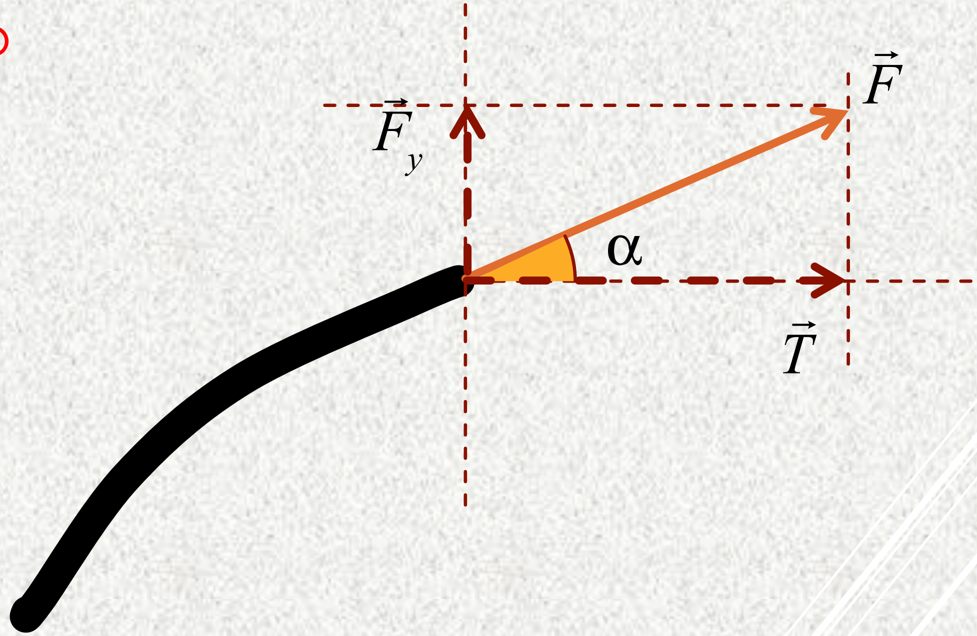
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(-v) \quad \text{e} \quad \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial^2 u} \cdot v^2$$

Portanto: $\dfrac{\partial^2 f}{\partial^2 u} = \dfrac{1}{v^2} \dfrac{\partial^2 f}{\partial t^2} = \dfrac{\partial^2 f}{\partial x^2}$

Equação da onda

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}}$$

Condições de contorno



$$\frac{\partial U}{\partial x} = \tan\alpha \quad F_y = T \cdot \tan\alpha$$

$$\mathcal{P}_{\text{otência}} = F \cdot v = F_y \cdot \dot{U}$$

$$\dot{U} = 0 \Rightarrow U\left(x_b, t\right) = \text{constante} \quad \left(\text{Dirichlet}\right)$$

$$\mathcal{P}_{\text{otência}} = 0, \text{ mas o extremo não é fixo } \Rightarrow \frac{\partial U\left(x_b, t\right)}{\partial x} = 0 \left(\text{Neumann}\right)$$

Resolver numericamente:

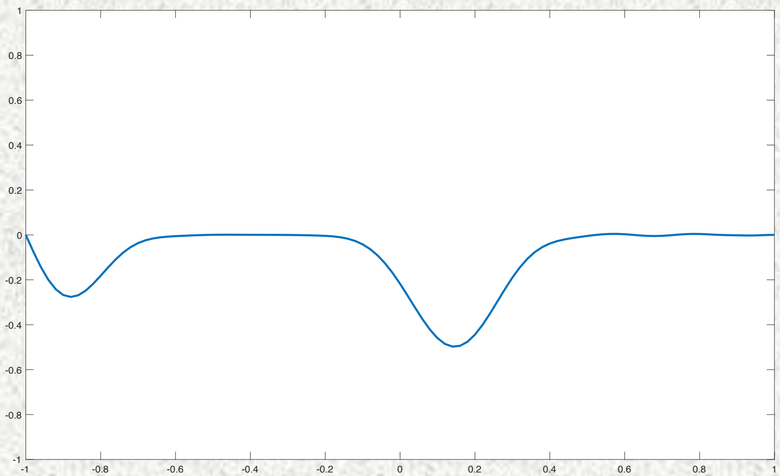
I) Série de Chebyshev (grau ~ 25 p/ x) e LeapFrog p/ t $\left(dt = 4/N^2\right)$

$$\frac{\partial^2 U(x,t)}{\partial t^2} = 1 \cdot \frac{\partial^2 U(x,t)}{\partial x^2},$$

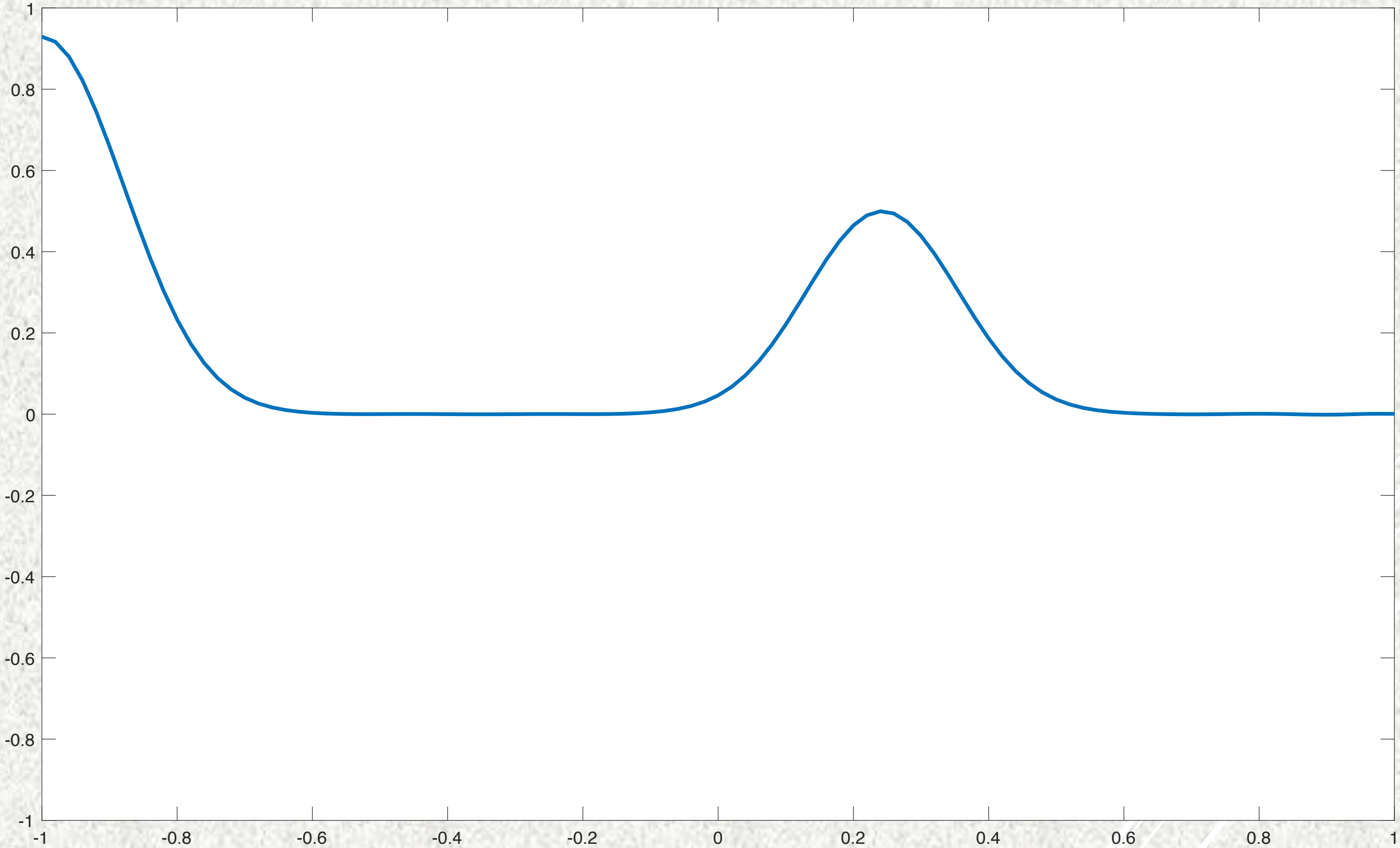
com $U(x,0) = \exp\left(-40(x-0.4)^2\right), \quad x \in [-1,1] \subset \mathbb{R}$ e $t \in [0,3]$.

1^a resolução: extremos fixos

2^a resolução: extremos livres



Wave1D.m
Wave1Dfree.m
Wave1Dfree2.m



Menor discretização:

$$\Delta x = 1 - \cos\left(\frac{1 \cdot \pi}{N}\right) = 2 \cdot \sin^2\left(\frac{\pi}{2N}\right) \simeq \frac{\pi^2}{2N^2} \simeq \frac{5}{N^2}$$

```
function  Wave1D(N)

D2 = diffmat(N+1,2); dt = 4/N^2;

xs = chebpts(N+1);

U = exp(-40*(xs-0.4).^2); % Gaussian pulse
U([1,N+1]) = 0; % Tight extrems

Uold = U; t = 0;
%%

kmax = round(4/dt);

for k = 0:kmax

    Uxx = D2*U;

    Unew = 2*U - Uold + dt^2*(Uxx);%Leap frog
    Uold = U; U = Unew; t = t+dt;
    U([1,N+1]) = 0; % Tight extrems
        %% Spectral plot
        xp = linspace(-1,1,101).';
        Uplot = bary_Berrut(xs,U,xp);


            plot(xp,Uplot,'LineWidth',3);
            axis([-1 1 -1 1]); title(num2str(t,3))
            set(gcf,'position',[0 200 1400 800]);
            pause(0.1)


end
```

```
function  Wave1Dfree2(N)

D2 = diffmat(N+1,2); dt = 4/N^2;
D1 = diffmat(N+1,1);

xs = chebpts(N+1);

U = exp(-40*(xs-0.4).^2); U([1,N+1]) = 0;
Uold = U;
%%
kmax = round(4/dt);

for k = 0:kmax

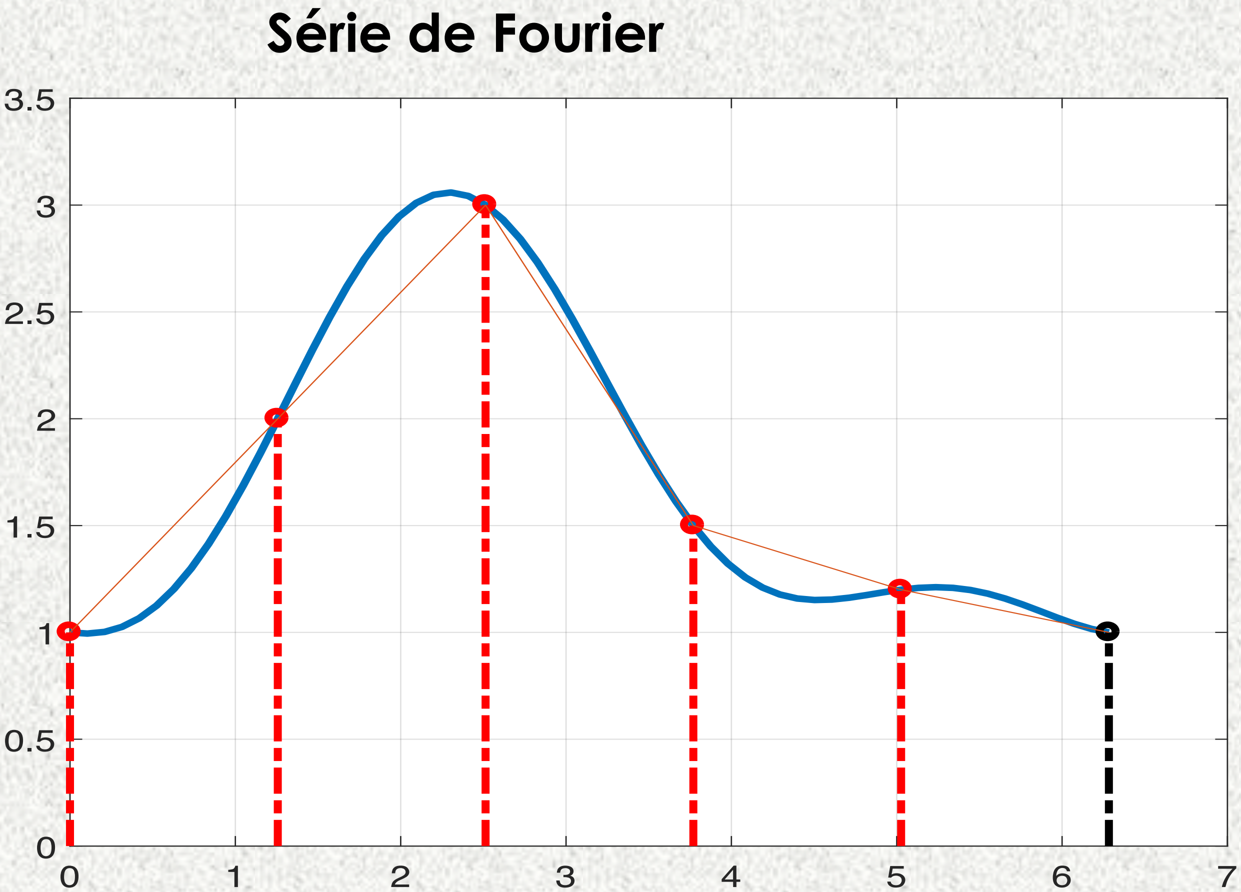
    Ux = D1*U; Ux([1,N+1],:) = 0;%Neumann - Free ends
    Uxx = D1*Ux;

    Unew = 2*U - Uold + dt^2*(Uxx);%Leap frog
    Uold = U; U = Unew;
        %% Spectral plot
        xp = linspace(-1,1,101).';
        Uplot = bary_Berrut(xs,U,xp);


            plot(xp,Uplot,'LineWidth',3);
            axis([-1 1 -1 1]);
            set(gcf,'position',[0 200 1400 800]);
            pause(0.1)


end
```

Tarefa
Escrever e plotar o código do slide anterior para uma corda que tem um extremo fixo e outro livre.



$$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

$$y_5 = y_0$$

$$I = \frac{(y_0 + y_1)}{2} \cdot \Delta x + \ldots + \frac{(y_4 + y_0)}{2} \cdot \Delta x. \quad \text{Cada } y_k \text{ aparece 2 vezes}$$

$$I = \frac{2\pi}{5}\left[y_1 + \ldots + y_5\right] = \frac{2\pi}{5}\left[y_0 + \ldots + y_4\right]$$

Séries de Fourier

Fourier series coefficients

$$A_0 = \frac{1}{P}\int_{-P/2}^{P/2} s(x)\,dx$$

$$A_n = \frac{2}{P}\int_{-P/2}^{P/2} s(x)\cos\left(\frac{2\pi nx}{P}\right)dx \qquad \text{for } n \geq 1$$

$$B_n = \frac{2}{P}\int_{-P/2}^{P/2} s(x)\sin\left(\frac{2\pi nx}{P}\right)dx \qquad \text{for } n \geq 1$$

Forma discreta de ordem m p/ $I_F = \left[0, 2\pi\right]$

$$A_n = \frac{1}{\pi}\sum_{j=0}^{2m} f\left(x_j\right)\cdot\cos\left(nx_j\right)\cdot\Delta x, \quad \text{e} \qquad \Delta x = \frac{2\pi}{2m+1} \qquad n \neq 0$$

$$A_n = \frac{2}{2m+1}\sum_{j=0}^{2m} f\left(x_j\right)\cdot\cos\left(nx_j\right) \qquad B_n = \frac{2}{2m+1}\sum_{j=0}^{2m} f\left(x_j\right)\cdot\sin\left(nx_j\right)$$

$$A_0 = \frac{1}{2m+1}\sum_{j=0}^{2m} f\left(x_j\right)\cdot 1$$

$$f\left(x\right) = A_0 + A_1\cos\left(x\right) + B_1\sin\left(x\right) + \ldots + B_n\sin(nx)$$

Base de Fourier discreta

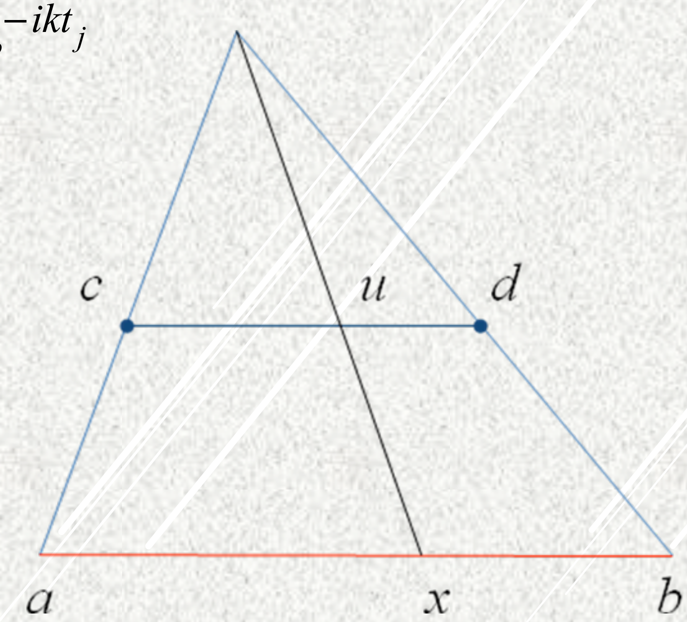
$$f(t) = \sum_{k=-m}^{m} c_k e^{ikt} \quad , \quad t \in [0, 2\pi]$$

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-ikt}\, dt \cong \frac{1}{2m+1}\sum_{j=0}^{2m} f\left(t_j\right) e^{-ikt_j}$$

Mudança de domínio

$$\frac{t-0}{2\pi} = \frac{x-a}{2L}, \quad \text{com } L = \frac{b-a}{2}$$

Assim, $t = \frac{\pi}{L}\left(x-a\right)$



$$T_k = \frac{2\pi}{k}, \quad f_k = \frac{k}{2\pi} \quad \left(\omega_k = k\right)$$

$$\text{Como } f(t) = \sum_{k=-m}^{m} c_k e^{ikt} = \left[e^{ik(-m:m)t} \right] \begin{bmatrix} c_{-m} \\ \vdots \\ c_0 \\ \vdots \\ c_m \end{bmatrix} = \left\langle B \middle| \hat{f} \right\rangle, \qquad D_{\mathrm{S}} = i \cdot \begin{bmatrix} -m & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & m \end{bmatrix}$$

$$\frac{df}{dt} = \sum_{k=-m}^{m} ik \cdot c_k e^{ikt} = \sum_{k=-m}^{m} e^{ikt} \cdot ik \cdot c_k = B \cdot \mathrm{diag}\left[1i(-m:m) \right] \cdot \hat{f} = \left\langle B \middle| D_{\mathrm{S}} \middle| \hat{f} \right\rangle$$

$$D_{\mathrm{Physical}} = B \cdot D_{\mathrm{S}} \cdot \left(B \right)^{-1} \qquad \frac{df_j}{dt} = D_{\mathrm{Phys}} \cdot f_j$$

Há uma expressão fechada p/ D_{Physical}.

```
% Bases vectors of Complex Fourier

function BF = Base_Comp_F(m,a,b)
%%
L = (b-a)/2;
xs = linspace(a,b,2*m+2).'; xs(end) = [];
BF = zeros(2*m+1);
    for k = -m:m
        BF(:,k+m+1) = exp(1i*pi*k*xs/L);
    end
```

```
% Inverse Bases vector of Complex Fourier

function invBF = invBase_Comp_F(m,a,b)
%%
L = (b-a)/2;
xs = linspace(a,b,2*m+2); xs(end) = [];
invBF = zeros(2*m+1);
    for k = -m:m
        invBF(k+m+1,:) = exp(-1i*pi*k*xs/L)/(2*m+1);
    end
```

```
>> h = 2*pi/N;
>> [0, .5*csc((1:N-1)*h/2)]';
>>
```

Physical D: ordem 1, N = 7.

```
D =

        0    1.1524   -0.6395    0.5129   -0.5129    0.6395   -1.1524
  -1.1524         0    1.1524   -0.6395    0.5129   -0.5129    0.6395
   0.6395   -1.1524         0    1.1524   -0.6395    0.5129   -0.5129
  -0.5129    0.6395   -1.1524         0    1.1524   -0.6395    0.5129
   0.5129   -0.5129    0.6395   -1.1524         0    1.1524   -0.6395
  -0.6395    0.5129   -0.5129    0.6395   -1.1524         0    1.1524
   1.1524   -0.6395    0.5129   -0.5129    0.6395   -1.1524         0
```

Fast Fourier Transform: fft

Tarefa

Deriv_Trig.jl

Bary_Trig.jl

Se $f(x) = a_0/2 + a_1\cos(x) + b_1\sin(x) + \ldots + b_m\sin(mx)$ e $f(x) = \sum_{k=-m}^{m} c_k e^{ikx}$

então:

$$\begin{cases} c_k = (a_k - i\cdot b_k)/2 \\ c_{-k} = (a_k + i\cdot b_k)/2 \end{cases} \text{ e } \begin{cases} a_k = c_k + c_{-k} \\ b_k = (c_k - c_{-k})\cdot i \end{cases} \text{ e } a_0 = 2c_0$$

Esse resultado independe de $f(x)$ ser real.

Ordem "natural": $C = \left[c_{-m}, c_{-m+1}, \ldots, c_0, c_1, \ldots, c_m\right]$

Ordem Matlab: $C = \left[c_0, c_1, \ldots, c_m, c_{-m}, c_{-m+1}, \ldots, c_{-1}\right]$ e

$$C = \left[c_0, c_1, \ldots, c_{M/2-1}, c_{-M/2}, c_{-M/2+1}, \ldots, c_{-1}\right] \text{ se } M \text{ for par.}$$

Resolver numericamente (Sol. Analítica: $y = \cos(\pi x)$

$$y'' + y' + \pi^2 y = -\pi \sin(\pi x) \quad x \in [-1,1],$$

nas seguintes condições:

I) $y(-1) = y(+1) = -1$ Dirichlet

II) $y'(-1) = y'(+1) = 0$ Neumann

III) $y(-1) + y'(-1) = -1$ Robin

$y(+1) + y'(+1) = -1$

```
%% Atividade 5 ====================
F = @(x) cos(pi*x); %Analytical solution
n = 30;   xp = -1+(0:100).'*2/100; ya = F(xp);
xs = cos( (0:n).'*pi/n );
D = Generalized_Diff_Mat(xs); D2 = D^2; II = eye(n+1);

%% Sistema para o miolo
xm = xs(2:n);
Sm = D2(2:n,:) + D(2:n,:) + pi*pi*II(2:n,:);
RHSm = -pi*sin(pi*xm);
%% Contorno
Sc = zeros(2,n+1); Sc(1,1) = 1; Sc(2,n+1) = 1;
RHSc = [-1; -1];
%% Gran finale
S = [Sm;Sc]; RHS = [RHSm;RHSc];
%%
y = S\RHS;

yp = bary_Berrut(xs,y,xp);
figure;
plot(xp,ya-yp);
grid on
```

```
%% Change the boundary codition
% Neumann at +1 and -1
Sc(1,:) = D(1,:); % x = -1
Sc(2,:) = D(n+1,:); % x = + 1
RHSc = [0;0];
RHS = [RHSm;RHSc]; S = [Sm;Sc];
y = S\RHS;

yp = bary_Berrut(xs,y,xp);
figure;
plot(xp,ya-yp);
grid on

%% Robin condition
% bc = a*y + b*y'
Ll = zeros(1,n+1); Lr = Ll;
Ll(1) = 1; Lr(n+1) = 1;
Sc(1,:) = Ll + D(1,:);
Sc(2,:) = Lr + D(n+1,:);
RHSc = [-1;-1];
RHS = [RHSm;RHSc]; S = [Sm;Sc];
y = S\RHS;

yp = bary_Berrut(xs,y,xp);
figure;
plot(xp,ya-yp);
grid on
```

Polinômios de Legendre

Sturm-Liouville

$$\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]+l(l+1)P_l(x)=0 \quad x\in\left[-1,1\right]$$

Ortogonalidade

$$\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]+l(l+1)P_l(x)=0 \qquad \times P_m$$

$$-\left\{\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]+m(m+1)P_m(x)\right\}=0 \qquad \times P_l$$

$$P_m(x)\frac{d}{dx}\left[\left(1-x^2\right)P_l'(x)\right]-P_l(x)\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]\ldots$$

$$\ldots+[l(l+1)-m(m+1)]P_m(x)P_l(x)=0$$

Os dois primeiros termos podem ser escritos como:

TAREFA

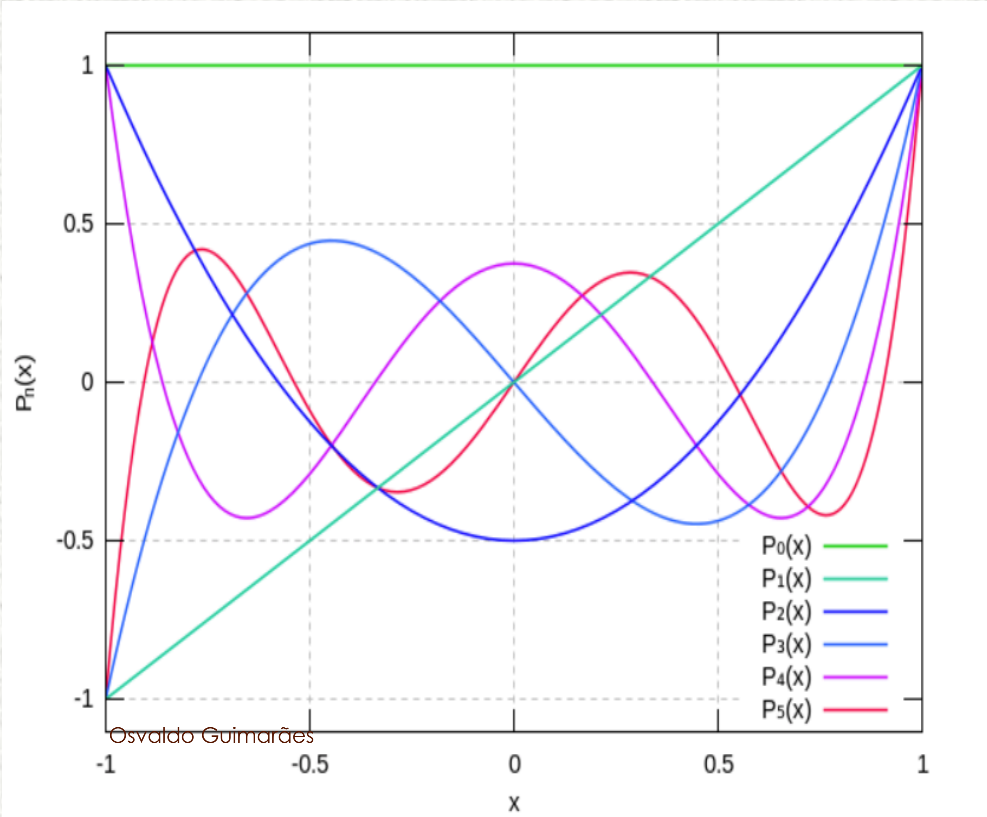
$$\frac{d}{dx}\left[\left(1-x^2\right)\left(P_mP_l'-P_lP_m'\right)\right],$$ cuja integral de -1 até 1 é nula.

$$[l(l+1)-m(m+1)]\int_{-1}^{1}P_m(x)P_l(x)dx=0$$

Portanto, se $m\neq l \Rightarrow \quad \int_{-1}^{1}P_m(x)P_l(x)dx=0$

Polinômios de Legendre

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn} \qquad x \in [-1,1] \subset \mathbb{R}$$



Osvaldo Guimarães

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

$$P_0 = 1,\ P_1 = x,\ \ P_n(1) = 1 \quad P_n(-1) = (-1)^n \quad \int_{-1}^{1} P_k(x)\cdot dx = 0\,(k \neq 0)$$

Recorrência: $(n+1)P_{n+1} = x(2n+1)P_n - nP_{n-1}$

$$P_n = \frac{P'_{n+1} - P'_n}{2n+1} \Rightarrow \int_{-1}^{x} P_n(u)\,du = \left.\frac{P_{n+1} - P_n}{2n+1}\right|_{-1}^{x}$$

Legendre como base completa no espaço de Hilbert

$$\lim_{n\to\infty}\int_a^b\left(f_{\text{Analytical}}(x)-\sum_{k=0}^{n}c_k\cdot P_k(x)\right)^2\cdot dx=0$$

$$f_n=\langle B_n|c_n\rangle$$

Cada c_k é obtido por: $c_k=\dfrac{2k+1}{2}\displaystyle\int_{-1}^{1}f\cdot P_k dx.$

Isto é, $c_k=\langle P_k|f\rangle\cdot\dfrac{2k+1}{2}$ (projetor): $\dfrac{?|P_k\rangle}{\langle P_k|P_k\rangle}.$

Interesse meramente teórico

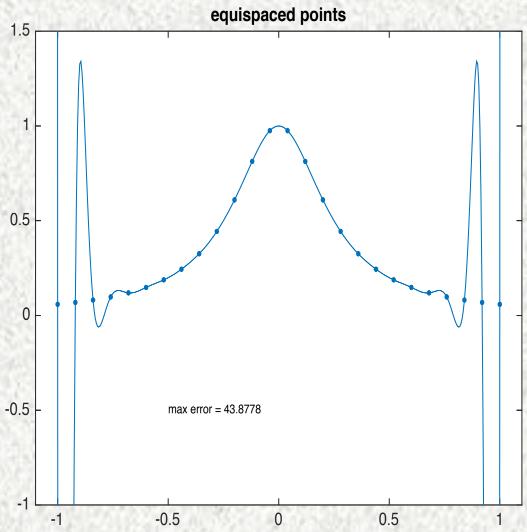
$$x^n=\sum_{k=n,n-2,\ldots}\frac{(2k+1)n!}{2^{(n-k)/2}\left(\frac{1}{2}(n-k)\right)!(k+n+1)!!}P_k(x)$$

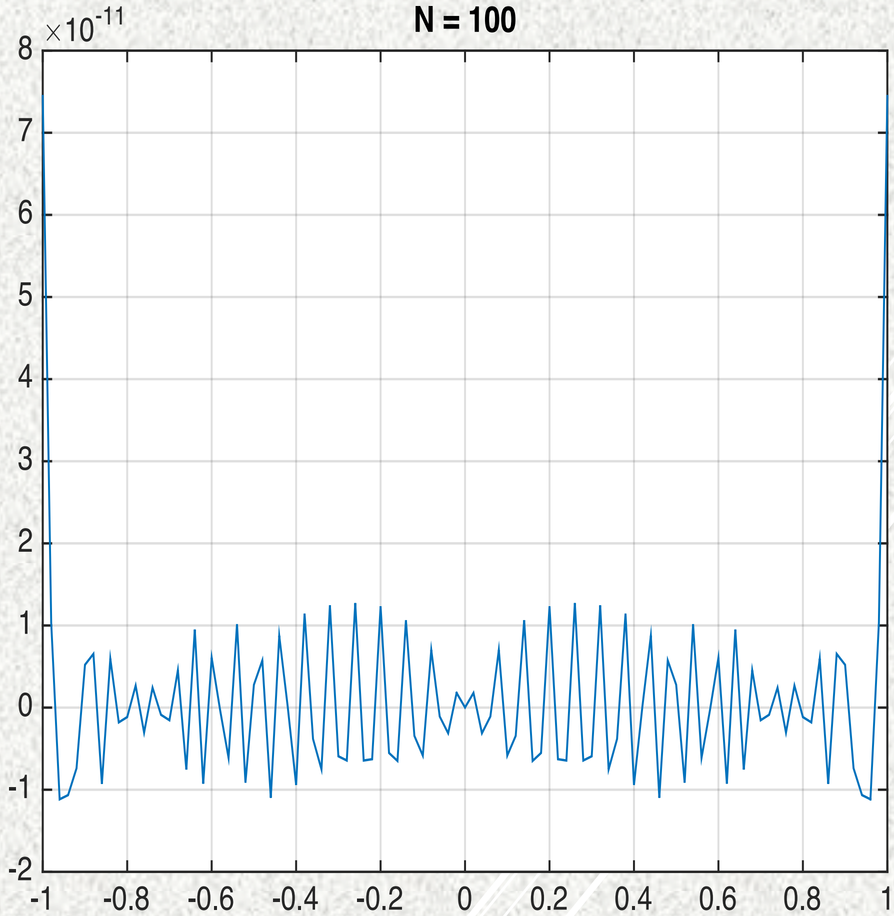
Qualquer P_k de Legendre é uma combinação linear de monômios.

Reciprocamente, qualquer polinômio de grau n é uma combinação linear de de $P_{k=0:n}$.

```
% Teste of Leg points - Chebfun
%%
N = 100;
Tb = legpoly(0:N);
[xs,ws] = legpts(N+1); Bx = Tb(xs);
f = @(x)1./(1+16*x.^2); c = zeros(N+1,1);
for k = 1:N+1
    c(k) = ws*(Bx(:,k).*f(xs))*(2*k-1)/2;
end
%% Plot results
xp = linspace(-1,1,101).'; yn = Tb(xp)*c;
plot(xp,yn-f(xp)); title(['N = ',num2str(N)]);
set(gca,'xtick',[-1:0.2:1]); grid on
```

Interpolação de: $f\left(x\right)=\dfrac{1}{1+16x^2}$



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```

$$I_n=\sum w_i\cdot f\left(x_i\right),\ \ i=0:n$$

TAREFA

1) Resolver numericamente (Sol. Analítica: $y = \cos(\pi x)$)

$$y'' + y' + \pi^2 y = -\pi \sin(\pi x) \quad x \in [-1,1],$$

nas seguintes condições, com $n = 31$, série de Chebyshev:

I) $y(-1) = y(+1) = -1$ Dirichlet

II) $y'(-1) = y'(+1) = 0$ Neumann

III) $y(-1) + y'(-1) = -1$ Robin

$y(+1) + y'(+1) = -1$ (slide 14)

2) Demonstrar a igualdade do slide 26 sobre ortogonalidade.