AULA 02
INTRODUÇÃO AOS MÉTODOS ESPECTRAIS
PTC 5725 (25/09/2025)

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Interpolação baricêntrica

Interpolação de Lagrange
Função polyfit no Matlab

$$w_j = \prod_{m\neq j}(x_j - x_m)^{-1}$$

$$\ell_j(x) = \frac{\prod_{k\neq j}(x - x_k)}{\prod_{k\neq j}(x_j - x_k)} \qquad L(x) = \sum_{j=0}^{n} \ell_j(x)\cdot f_j$$

Instabilidade

Se $x_j \approx x_k$, o erro relativo de $(x_j - x_k)^{-1}$

é expressivo!

$$L(x) = \sum_{j=0}^{k} \frac{w_j}{x - x_j} y_j \Bigg/ \sum_{j=0}^{k} \frac{w_j}{x - x_j}.$$

```matlab
%  The function p = polint(xk, fk, xnew) computes the polynomial interpolant
%  of the data (xk, fk).  Two or more data points are assumed.
%
%  Input  (constant weight)
%  xk:    Vector of x-coordinates of data (assumed distinct).
%  fk:    Vector of y-coordinates of data.
%  xnew:     Vector of x-values where polynomial interpolant is to be evaluated.
%

%  Output:
%  fkk:    Vector of interpolated values.
%  Mi: Matrix of interpolation

function [fnew,Mi] = Bary_Generic(xk, fk, xnew)

     xnew = xnew(:);                          % Make sure the data are column vectors
    xk = xk(:);  fk = fk(:);


     N = length(xk);
     L = logical(eye(N));

     D = xk(:,ones(1,N))-xk(:,ones(1,N))';  % Compute the weights w(k)
     D(L) = ones(N,1);
     w = 1./prod(D)';

%%
Mi = bsxfun(@minus,xnew,xk');          %All xnew(j) - x(k)
Mi = bsxfun(@rdivide,w',Mi);            %All w(k)/(xnew(j) - x(k))
Mi = bsxfun(@rdivide, Mi, sum(Mi,2));   % Normalization
Mi(isnan(Mi)) = 1;                      % Remove NaNs
%%

     fnew = Mi*fk;

end
```

$$f(x) = 10^3 \sin(\pi x) \quad x \in [-1,1]$$

- Aproxime *f* por um polinômio de grau 21(pontos de Cheby).
- Use interpolação de Lagrange p/ calcular *f* em 501 pontos equiespaçados.
- Compare com a interpolação baricêntrica.
- Compare com o resultado analítico.

Cheby Poly (I) T_n

$$x \in [-1,1] \qquad T_n(\cos\theta) = \cos(n\theta) \qquad x = \cos\theta$$
$$\theta \in [0,\pi]$$

$$T_0(x) = 1 \text{ e } \quad T_1(x) = x \qquad \frac{dx}{d\theta} = -\sin\theta$$

Ortogonalidade $\quad T_n(\cos\theta) = \cos(n\theta) = P_n(\theta)$

$$\int_0^\pi P_n \cdot P_m \, d\theta = \int_1^{-1} T_n \cdot T_m \cdot \frac{1}{-\sin\theta} dx = \int_{-1}^{1} T_n \cdot T_m \cdot \frac{1}{\sqrt{1-x^2}} dx$$

Portanto,

$$\langle T_n | T_m \rangle_w = \begin{cases} \pi/2, \text{ se } \; m = n \neq 0 \\ \pi \quad \text{ se } \; m = n = 0 \\ 0 \;\; \text{se } m \neq n \end{cases}$$

$$\cos a \cdot \cos b = [\cos(a-b) + \cos(a+b)]/2$$

$$\cos(m\theta) \cdot \cos(n\theta) = [\cos((m-n)\theta) + \cos((m+n)\theta)]/2$$

Se $\quad m = n = 0 \Rightarrow \int_0^\pi 1 \cdot d\theta = \pi.$

Se $\quad m = n \neq 0 \Rightarrow \int_0^\pi 1 \cdot d\theta/2 + \underbrace{\int_0^\pi \cos(k\theta) \cdot d\theta/2}_{0} = \pi/2 \quad k \in \mathbb{N}^*$

Se $\quad m \neq n \quad \Rightarrow \underbrace{\int_0^\pi \cos(k_1\theta) \cdot d\theta/2}_{0} + \underbrace{\int_0^\pi \cos(k_2\theta) \cdot d\theta/2}_{0} = 0$

Cheby Series

$$f(x) \approx \sum{}' a_k T_k \qquad\qquad a_k = \frac{2}{\pi}\int_{-1}^{1} f \cdot T_k \frac{dx}{\sqrt{1-x^2}} = \frac{\langle f | T_k \rangle}{\langle T_k | T_k \rangle}$$

$$\text{Operador projetor: proj} = \frac{|T_k\rangle}{\langle T_k | T_k \rangle}$$

Lobatto points: $\theta_i = i \cdot \pi/n, \qquad i = n:-1:0$

$$x_i = \cos\theta_i$$

Matriz de Base ordem n

$$B_{i+1,j+1} = \cos\left(j\theta_i\right) = T_j\left(x_i\right)$$

$$B = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \dots & T_n(x_0) \\ T_0(x_1) & \dots & & \\ \vdots & & \ddots & \\ T_0(x_n) & & & T_n(x_n) \end{bmatrix}$$

"Inversa" da Base n

$$BI = \frac{1}{n}\begin{bmatrix} T_0(x_0)/2 & T_0(x_1) & \dots & T_0(x_n)/2 \\ 2T_1(x_0) & \dots & & 2T_1(x_n) \\ \vdots & & \ddots & \\ T_n(x_0)/2 & T_n(x_1) & \dots & T_n(x_n)/2 \end{bmatrix}$$

Transformação Discreta $\left(\text{Exata p/ polinômios até grau } n\right)$

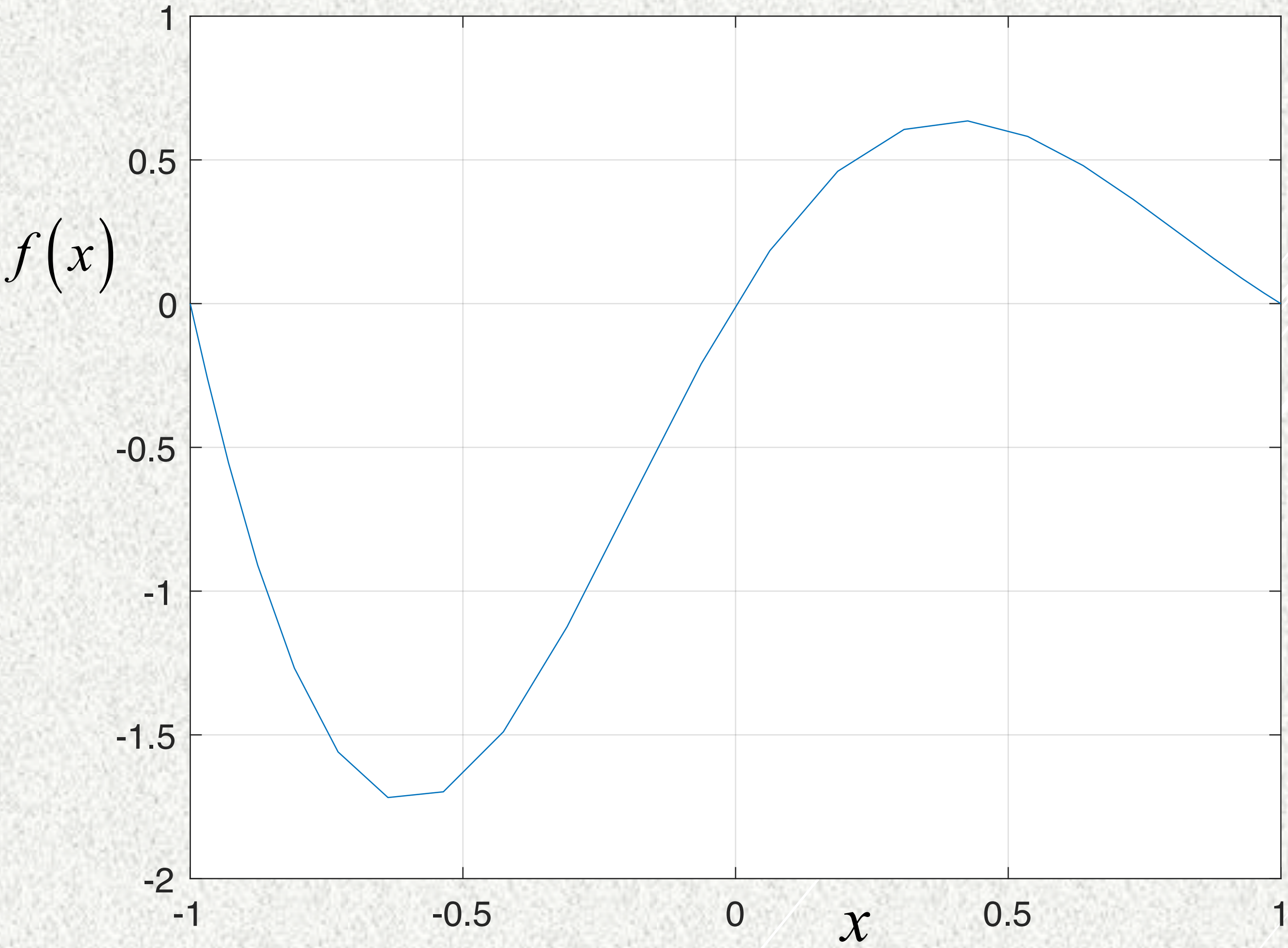
$$c_k = \frac{\alpha_k}{n}\sum{}'' T_k\left(x_j\right)p\left(x_j\right),$$

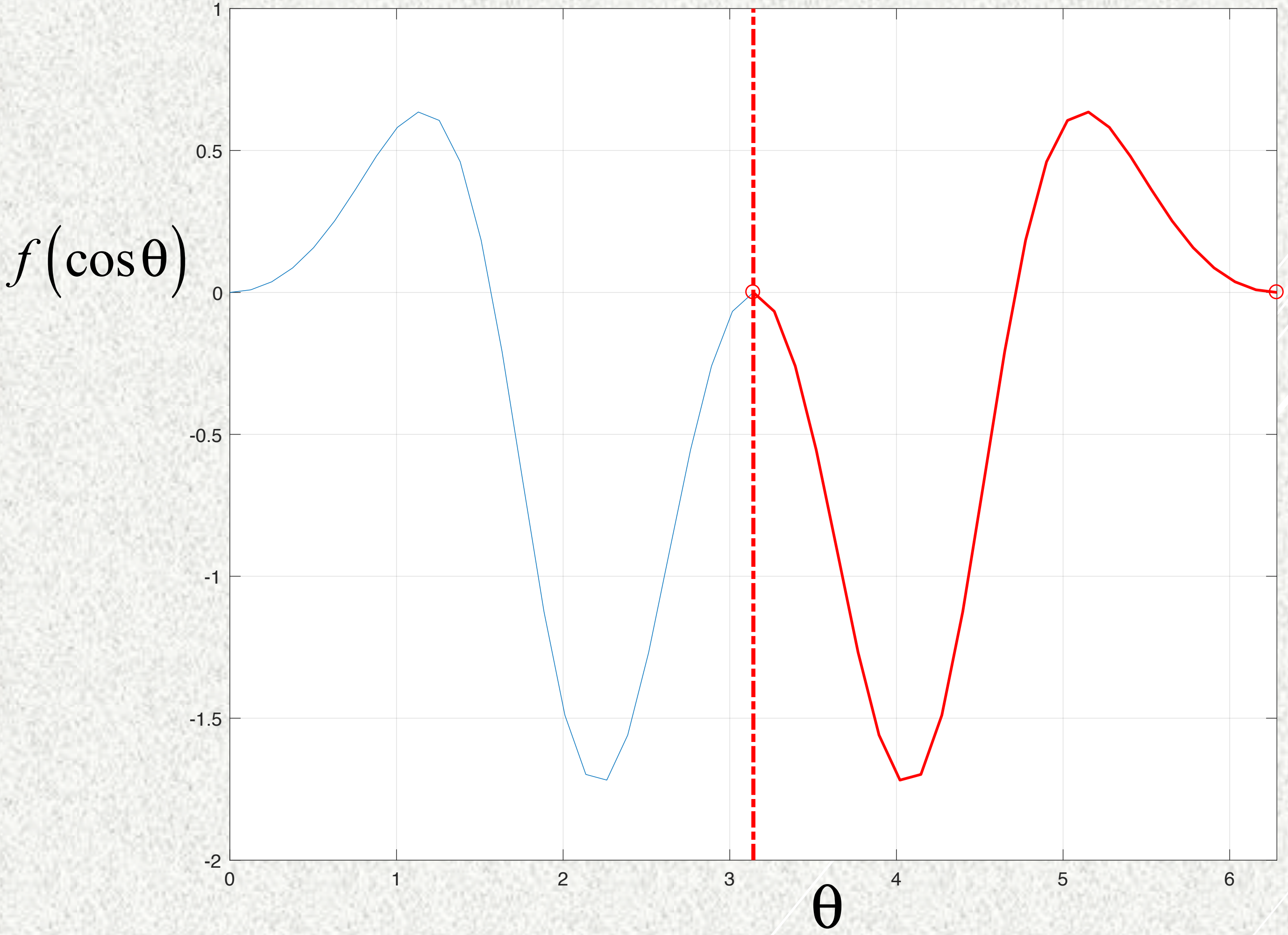
$\alpha_k = 1$ p/ $k = 0$ ou $k = n$ e $\alpha_k = 2$ p/ o resto.

Supondo $f(x)$ polinomial de grau $\alpha \le n$

$$f(x) = \left\langle B \middle| \mathbf{A} \right\rangle_n \qquad A = B^{-1} \cdot \mathbf{y}$$

$$f(x) = e^{-x}\sin(\pi x) \qquad n = 25$$

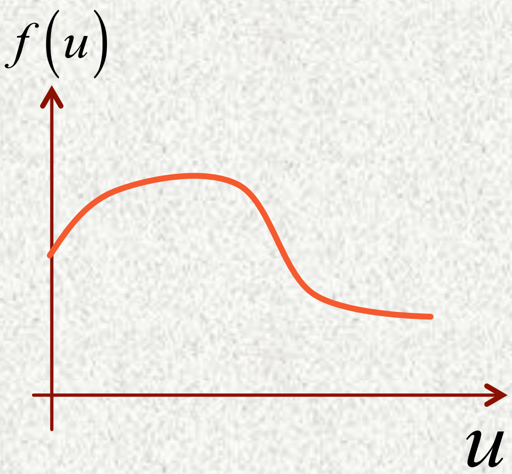


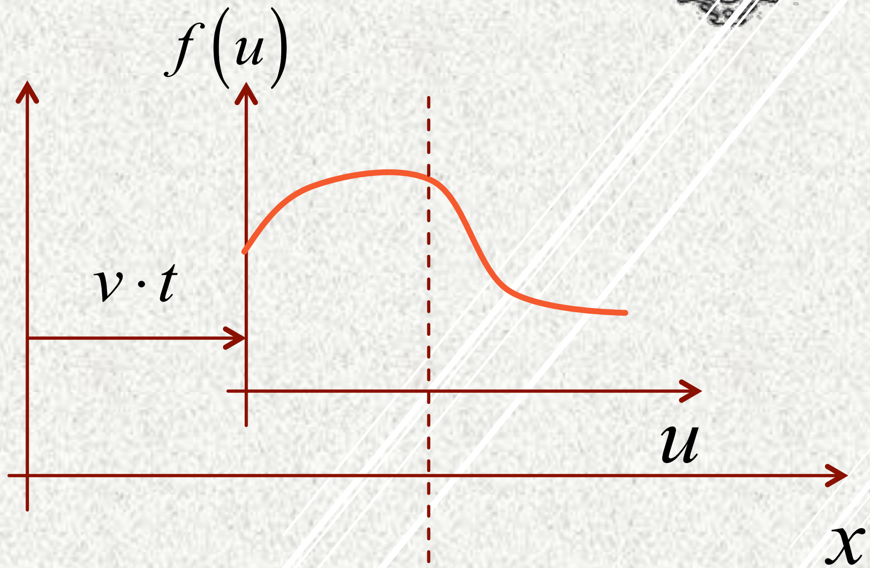


$$\cos\left(\pi-\theta\right)=\cos\left(\pi+\theta\right)$$

```
function B = FOU_CHB(A,direction)
% A - original data in columns
% B - transformed data in columns
% direction - set equal to 1 for nodal to spectral
%             anything else for spectral to physical


[N,M]=size(A);

if direction==1 % physical-to-spectral

    A = flipud(A);% TO FOLLOW THETA DIRECTION

    F=ifft([A(1:N,:);A(N-1:-1:2,:)]);

    B=([F(1,:); 2*F(2:(N-1),:); F(N,:)]);



else            % Spectral-to-physical

     F=fft([A(1,:); [A(2:N-1,:);2*A(N,:);A(N-1:-1:2,:)]/2]);
     B = F(N:-1:1,:);


end
```

Equação da onda





Para cada abscissa u, temos: $f(u) = f(x - vt)$

$x = u + v \cdot t \Rightarrow u = x - vt$

$\dfrac{\partial u}{\partial x} = 1$ e $\dfrac{\partial u}{\partial t} = -v$

$$\boxed{\frac{\partial f}{\partial t} = -v \cdot \frac{\partial f}{\partial x}}$$

$\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u} \cdot \dfrac{\partial u}{\partial x} = \dfrac{\partial f}{\partial u}$ $\qquad$ $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial u} \cdot \dfrac{\partial u}{\partial t} = \dfrac{\partial f}{\partial u}(-v)$

Portanto:

$$\frac{\partial f}{\partial t} = -v \cdot \frac{\partial f}{\partial x}$$

Onda regressiva: $f = f(x + v \cdot t)$

Em ambos os casos, f é arbitrária.

Série de Fourier

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}$$

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx.$$

Para cada abscissa u, temos: $f(u) = f(x - vt)$

$x = u + v \cdot t \Rightarrow u = x - vt$

$$\frac{\partial u}{\partial x} = 1 \quad \text{e} \quad \frac{\partial u}{\partial t} = -v$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial^2 u}$$

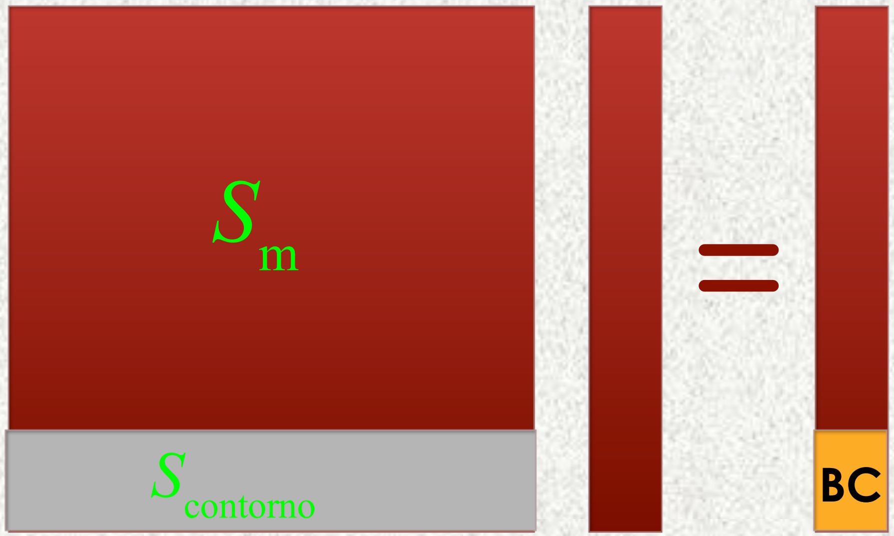
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(-v) \quad \text{e} \quad \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial^2 u} \cdot v^2$$

Portanto: $\dfrac{\partial^2 f}{\partial^2 u} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2} = \dfrac{\partial^2 f}{\partial x^2}$

Equação da onda

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

Collocation



$$S \cdot \vec{y} = \vec{r}$$

Boundary Value Problems (BVP)

Programa 13

$$y_{xx} = e^{4x}, \quad y(\pm 1) = 0$$

Analiticamente: $y = \left[e^{4x} - x \cdot \sinh(4) - \cosh(4) \right] / 16.$

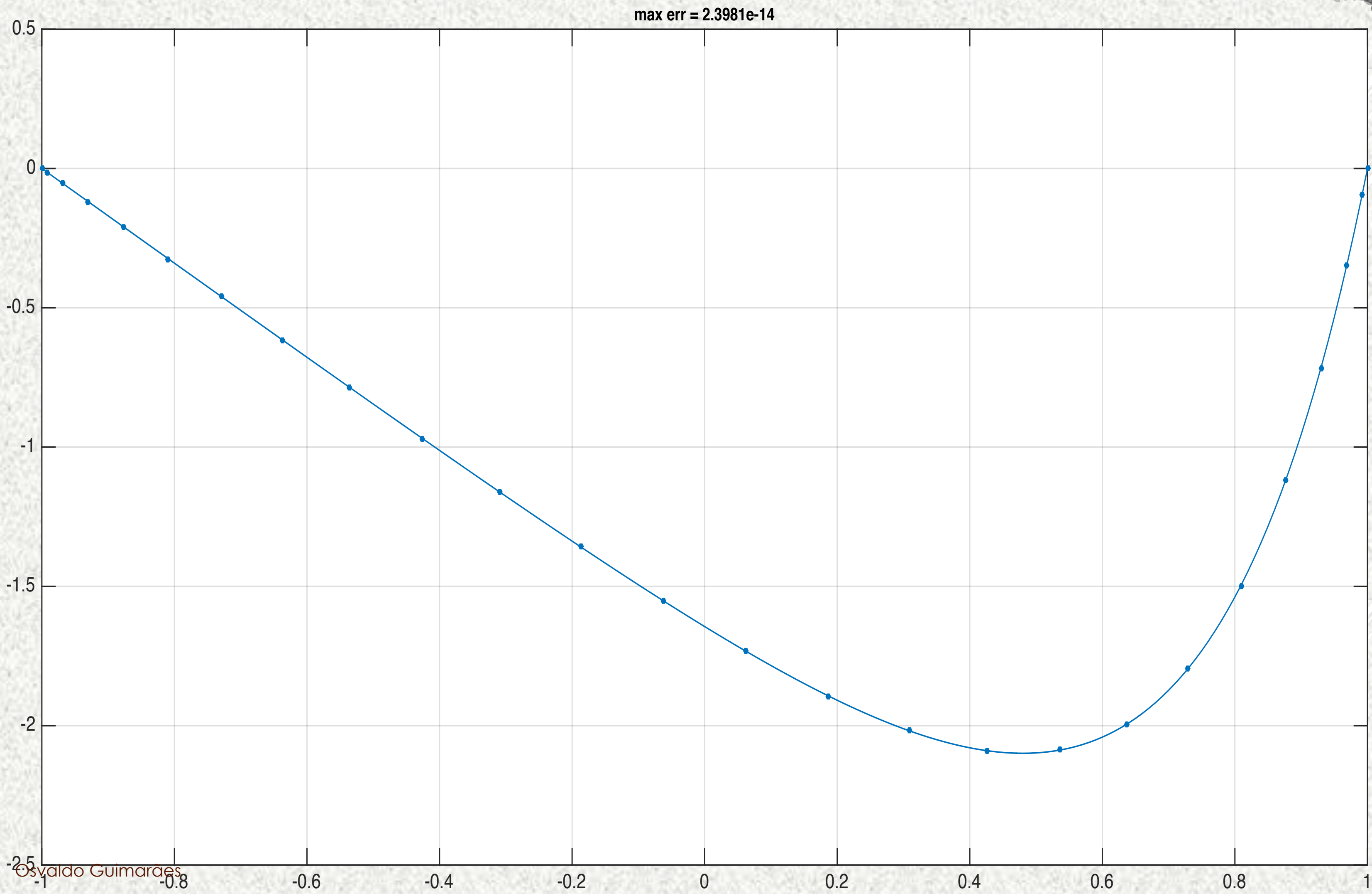
```
  %% BVP p.13 - Alternative code
N = 25;
x = -cos( (0:N)*pi/N ).'; D = Generalized_Diff_Mat(x); D2 = D^2;
D2([1,N+1],:) = 0; D2(1,1) = 1; D2(N+1,N+1) = 1;
f = exp(4*x(2:N)); f = [0;f;0];
u = D2\f;
```

Equação matricial

$$D \cdot (D \cdot y) = f(x),$$

$$D^2 \cdot y = f,$$

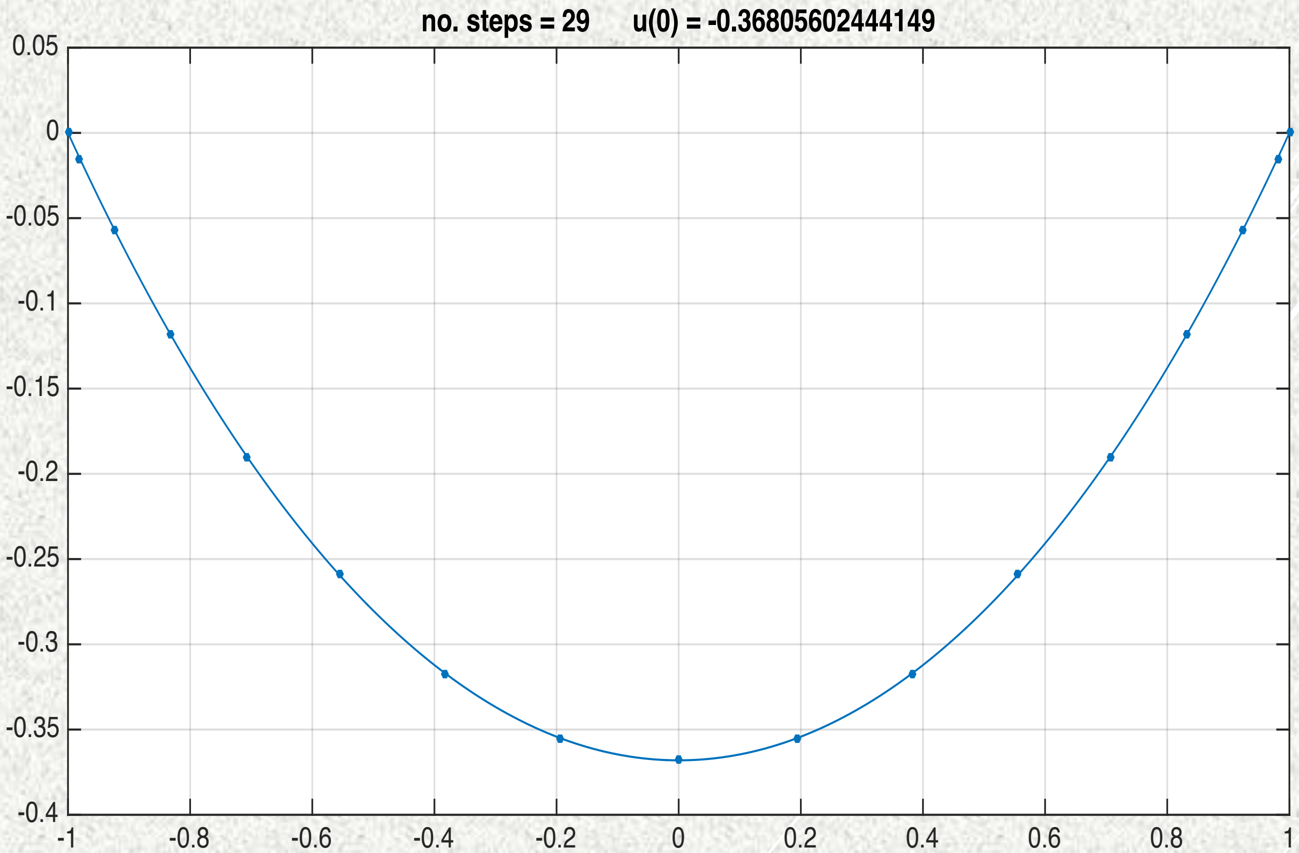
- Linha 1 de D: [1 0 ...0] e $f_1 = y(-1) = 0$
- Linha $(n+1)$ de D: $\left[0......1\right]$ e $f_{n+1} = y(+1) = 0$


max err = 2.3981e-14
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Problema 14 – BVP não linear

$$y_{xx} = e^{y}, \quad y(\pm 1) = 0.$$

$$\text{error} \cong \text{eps}$$



$$y_{xx} = -k^2 y, \quad y(\pm 1) = 0.$$

```
%% Alternative code
  N = 40;
x = -cos( (0:N)*pi/N ).'; D = Generalized_Diff_Mat(x); D2 = D^2;
D2([1,N+1],:) = 0; D2(:,[1,N+1])=0 ;
[V,lam] = eig(D2,'vector');
[lam,ii] = sort(-lam(lam<0)); % sort eigenvalues and -vectors
 V = V(:,ii);
```

Equação matricial

$$D^2 \cdot y = -k^2 y,$$

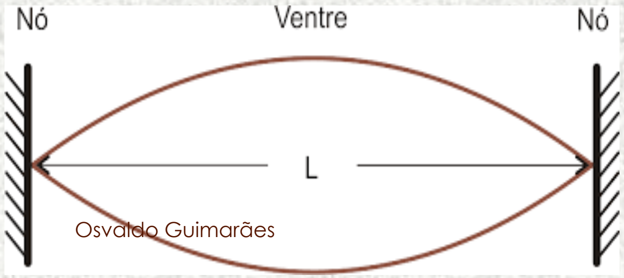
$$y(\pm 1) = 0$$

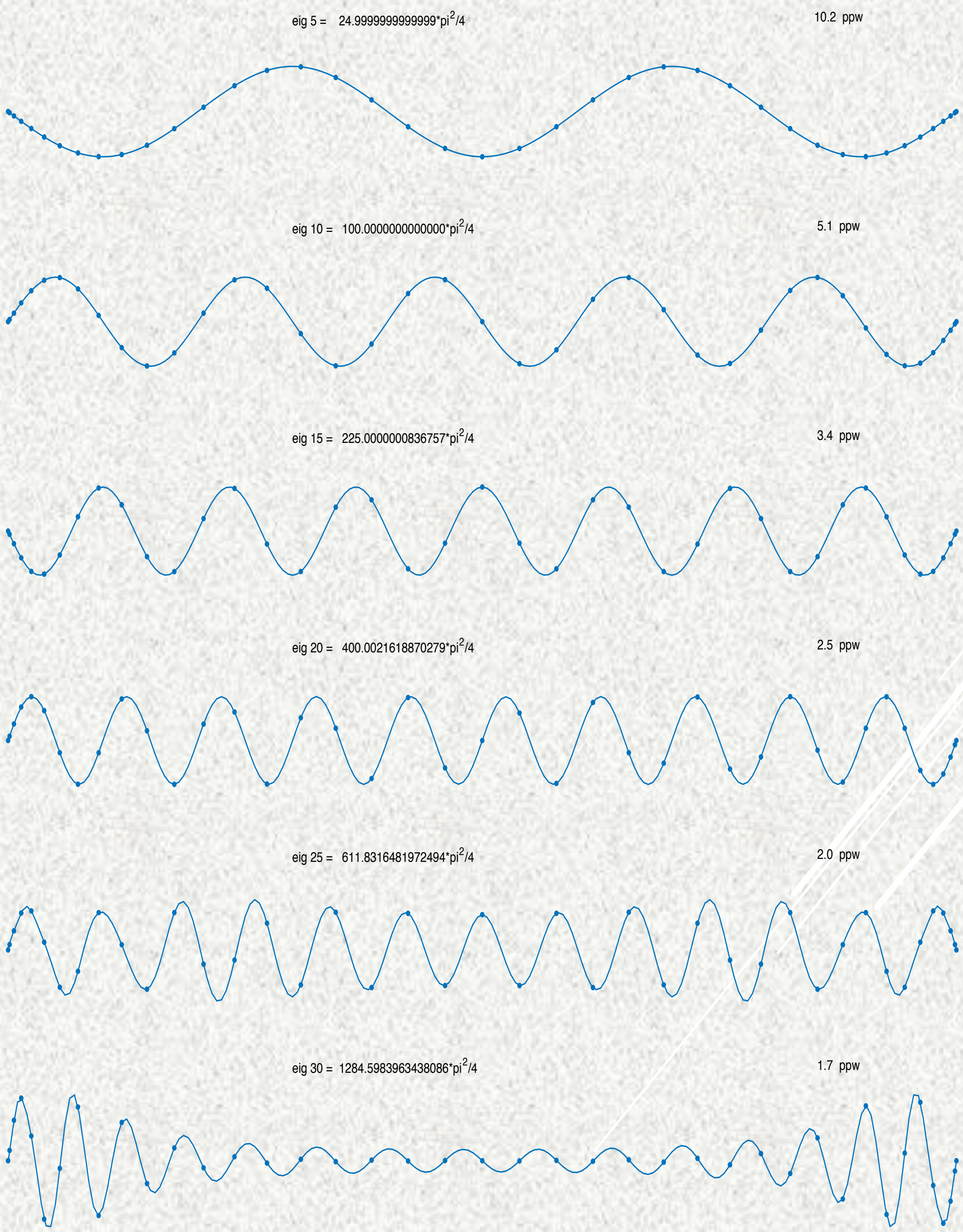
Soluções estacionárias c/ período máximo 4 $(2L)$.

$$\omega_1 = \frac{2\pi}{4} = \pi/2 \text{ rad/s} \qquad \omega_n = n \cdot \omega_1.$$

Autovalores

$$k_n^2 = \left(n \frac{2\pi}{T}\right)^2 = n^2 \frac{\pi^2}{4}$$





eig 5 = 24.9999999999999*pi^2/4
10.2 ppw
eig 10 = 100.0000000000000*pi^2/4
5.1 ppw
eig 15 = 225.0000000836757*pi^2/4
3.4 ppw
eig 20 = 400.0021618870279*pi^2/4
2.5 ppw
eig 25 = 611.8316481972494*pi^2/4
2.0 ppw
eig 30 = 1284.5983963438086*pi^2/4
1.7 ppw

TAREFA

I) Código para montar as matrizes do slide 7, dado n, com

$x = -\cos\left(k\pi/n\right),\ k = 0:n$

II) Código para obter os coeficientes de Cheby usando FFT, a partir dos valores da função nos nodos, conforme slide 10.

III) Resolver numericamente:

Série de Chebyshev (grau 7 ou 8):

$$\frac{x^2}{20}y'' + x\cdot y' - y - 5x^5 + 1 = 0,\ \text{com } x \in \left[-1,1\right] \quad y\left(-1\right) = 2 \text{ e } y\left(1\right) = 0$$

Testar todas as resoluções e enviar em pdf.