

Interpolating Normal Splines

Consider the following interpolation problem:

Problem: Given points $x_1 < x_2 < \dots < x_{n_1}$, $s_1 < s_2 < \dots < s_{n_2}$ and $t_1 < t_2 < \dots < t_{n_3}$ find a function f such that

$$\begin{aligned} f(x_i) &= u_i, \quad i = 1, 2, \dots, n_1, \\ f'(s_j) &= v_j, \quad j = 1, 2, \dots, n_2, \\ f''(t_k) &= w_k, \quad k = 1, 2, \dots, n_3, \\ n_1 &> 0, \quad n_2 \geq 0, \quad n_3 \geq 0. \end{aligned} \tag{1}$$

Note that knots $\{x_i\}$, $\{s_j\}$ and $\{t_k\}$ may coincide.

We assume that function f is an element of Hilbert space $H = H(X)$ over a set X (where X is R or an interval from R) and Hilbert space is selected in a such way that it is continuously embedded in space $C^2(X)$ of functions continuous with their second derivatives, therefore functionals F_i , F'_j , and F''_k

$$\begin{aligned} F_i(\varphi) &= \varphi(x_i), \quad F'_j(\varphi) = \varphi'(s_j), \quad F''_k(\varphi) = \varphi''(t_k), \quad \forall \varphi \in H, \\ x_i, s_j, t_k &\in X, \\ i &= 1, 2, \dots, n_1, \quad j = 1, 2, \dots, n_2, \quad k = 1, 2, \dots, n_3. \end{aligned}$$

are linear continuous functionals in H . It is obvious that all these functionals are linear independent. In accordance with Riesz representation theorem [1] these linear continuous functionals can be represented in the form of inner product of some elements $h_i, h'_j, h''_k \in H$ and $\varphi \in H$, for any $\varphi \in H$:

$$\begin{aligned} f(x_i) &= F_i(\varphi) = \langle h_i, \varphi \rangle_H, \quad F'_j(\varphi) = \langle h'_j, \varphi \rangle_H, \quad F''_k(\varphi) = \langle h''_k, \varphi \rangle_H, \quad \forall \varphi \in H, \\ i &= 1, 2, \dots, n_1, \quad j = 1, 2, \dots, n_2, \quad k = 1, 2, \dots, n_3. \end{aligned}$$

Elements h_i, h'_j and h''_k are twice continuously differentiable functions. Thereby the original system of constraints (1) can be written in form:

$$\begin{aligned} f(x_i) &= F_i(f) = \langle h_i, f \rangle_H = u_i, \\ f'(s_j) &= F'_j(f) = \langle h'_j, f \rangle_H = v_j, \\ f''(t_k) &= F''_k(f) = \langle h''_k, f \rangle_H = w_k, \\ f &\in H, \\ i &= 1, 2, \dots, n_1, \quad j = 1, 2, \dots, n_2, \quad k = 1, 2, \dots, n_3, \end{aligned} \tag{2}$$

here all functions $h_i, h'_j, h''_k \in H$ are linear independent and system of constraints (2) defines a nonempty convex and closed set (as an intersection of hyper-planes in the Hilbert space H).

Problem of reconstruction of function f satisfying system of constraints (2) is undetermined. We reformulate it as a problem of finding solution of this system of constraints that has minimal norm:

$$\sigma = \arg \min \{ \|f - z\|_H : (2), z \in H, \forall f \in H \}, \tag{3}$$

where $z \in H$ is a "prototype" function. Solution of this problem exists and it is unique ([6], [16]) as a projection of element z on the nonempty convex closed set in Hilbert space H . Element σ is an interpolating normal spline.

In accordance with generalized Lagrange method ([13], [16]) solution of the problem (3) can be presented as:

$$\sigma = z + \sum_{i=1}^{n_1} \mu_i h_i + \sum_{j=1}^{n_2} \mu'_j h'_j + \sum_{k=1}^{n_3} \mu''_k h''_k, \quad (4)$$

where coefficients μ_i , μ'_j and μ''_k are defined by system of linear equations

$$\begin{aligned} \sum_{l=1}^{n_1} g_{il} \mu_l + \sum_{j=1}^{n_2} g'_{ij} \mu'_j + \sum_{k=1}^{n_3} g''_{ik} \mu''_k &= u_i - \langle h_i, z \rangle_H, \quad 1 \leq i \leq n_1, \\ \sum_{i=1}^{n_1} g'_{ij} \mu_i + \sum_{l=1}^{n_2} g''_{jl} \mu'_l + \sum_{k=1}^{n_3} g'''_{jk} \mu''_k &= v_j - \langle h'_j, z \rangle_H, \quad 1 \leq j \leq n_2, \\ \sum_{i=1}^{n_1} g''_{ik} \mu_i + \sum_{j=1}^{n_2} g'''_{jk} \mu'_j + \sum_{l=1}^{n_3} g^{iv}_{kl} \mu''_l &= w_k - \langle h''_k, z \rangle_H, \quad 1 \leq k \leq n_3, \end{aligned} \quad (5)$$

Matrix of system (5) is the positive definite symmetric Gram matrix of the set of linearly independent elements $\{h_i\}$, $\{h'_j\}$, $\{h''_k\}$, and coefficients g_{il} , g'_{ij} , g''_{ik} , g'''_{jk} , g^{iv}_{kl} are defined as follows:

$$\begin{aligned} g_{il} &= \langle h_i, h_l \rangle_H, \quad g'_{ij} = \langle h_i, h'_j \rangle_H, \quad g''_{ik} = \langle h_i, h''_k \rangle_H \\ g'''_{jl} &= \langle h'_j, h'_l \rangle_H, \quad g'''_{jk} = \langle h'_j, h''_k \rangle_H, \quad g^{iv}_{kl} = \langle h''_k, h''_l \rangle_H. \end{aligned} \quad (6)$$

Let $H = H(X)$ be a reproducing kernel Hilbert space with reproducing kernel $V(\eta, \xi)$. Recall the definition of the reproducing kernel ([4], [7]). The reproducing kernel is a such function $V(\eta, \xi)$ that

- for every $\xi \in X$, $V(\eta, \xi)$ as function of η belongs to H
- for every $\xi \in X$ and every function $\varphi \in H$

$$\varphi(\xi) = \langle V(\eta, \xi), \varphi(\eta) \rangle_H \quad (7)$$

Reproducing kernel is a symmetric function:

$$V(\eta, \xi) = V(\xi, \eta),$$

also, in the considered here case it is twice continuously differentiable function by ξ and by η . Differentiating the identity (7) allows to get the identities for derivatives:

$$\frac{d\varphi(\xi)}{d\xi} = \left\langle \frac{\partial V(\cdot, \xi)}{\partial \xi}, \varphi \right\rangle_H, \quad \frac{d^2\varphi(\xi)}{d\xi^2} = \left\langle \frac{\partial^2 V(\cdot, \xi)}{\partial \xi^2}, \varphi \right\rangle_H. \quad (8)$$

Now it is possible to express functions h_i , h'_j , h''_k via the reproducing kernel V . Comparing (2) with (7) and (8) we receive:

$$\begin{aligned} h_i(\eta) &= V(\eta, x_i), \quad i = 1, 2, \dots, n_1 \\ h'_j(\eta) &= \frac{\partial V(\eta, s_j)}{\partial \xi}, \quad j = 1, 2, \dots, n_2, \\ h''_k(\eta) &= \frac{\partial^2 V(\eta, t_k)}{\partial \xi^2}, \quad k = 1, 2, \dots, n_3. \end{aligned} \quad (9)$$

The coefficients (6) of the Gram matrix can be presented as ([7], [8], [10]):

$$\begin{aligned} g_{il} &= \langle h_i, h_l \rangle_H = \langle V(\cdot, x_i), V(\cdot, x_l) \rangle_H = V(x_i, x_l), \\ g'_{ij} &= \langle h_i, h'_j \rangle_H = \left\langle V(\cdot, x_i), \frac{\partial V(\cdot, s_j)}{\partial \xi} \right\rangle_H = \frac{\partial V(x_i, s_j)}{\partial \xi}, \\ g''_{ik} &= \langle h_i, h''_k \rangle_H = \left\langle V(\cdot, x_i), \frac{\partial^2 V(\cdot, t_k)}{\partial \xi^2} \right\rangle_H = \frac{\partial^2 V(x_i, t_k)}{\partial \xi^2}. \end{aligned} \quad (10)$$

With the help of (7) and (10), we can also calculate g''_{jl} ([8], [10]):

$$\begin{aligned} g''_{jl} &= \langle h'_j, h'_l \rangle_H = \left\langle \frac{\partial V(\cdot, s_j)}{\partial \xi}, \frac{\partial V(\cdot, s_l)}{\partial \xi} \right\rangle_H = \\ &= \frac{d}{d\xi} \left\langle V(\cdot, \xi), \frac{\partial V(\cdot, s_l)}{\partial \xi} \right\rangle_H \Big|_{\xi=s_j} = \\ &= \frac{d}{d\xi} \left(\frac{\partial V(\xi, s_l)}{\partial \xi} \right) \Big|_{\xi=s_j} = \frac{\partial^2 V(s_j, s_l)}{\partial \eta \partial \xi}. \end{aligned} \quad (11)$$

Further

$$\begin{aligned} g'''_{jk} &= \langle h'_j, h''_k \rangle_H = \frac{\partial^3 V(s_j, t_k)}{\partial \eta \partial \xi^2}, \\ g^{\text{iv}}_{kl} &= \langle h''_k, h''_l \rangle_H = \frac{\partial^4 V(t_k, t_l)}{\partial \eta^2 \partial \xi^2}, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \langle h_i, z \rangle_H &= \langle V(\cdot, x_i), z \rangle_H = z(x_i), \\ \langle h'_j, z \rangle_H &= z'(s_j), \\ \langle h''_k, z \rangle_H &= z''(t_k). \end{aligned} \quad (13)$$

Normal splines will be constructed in Sobolev space $W_2^3[a, b]$ and in Bessel potential space $H_\varepsilon^3(R)$ (See [1, 2, 4, 20] for details). Elements of these spaces can be treated as twice continuously differentiable functions.

Reproducing kernel for Sobolev spaces $W_2^l[0, 1]$ (here l — any positive integer) was constructed in work [8]. Thus, reproducing kernel for Sobolev space $W_2^3[0, 1]$ with norm

$$\|f\|_{W_2^3} = \left(\sum_{i=0}^2 (f^{(i)}(0))^2 + \int_0^1 (f^{(3)}(s))^2 ds \right)^{1/2},$$

can be presented as

$$V(\eta, \xi) = \begin{cases} \sum_{i=0}^2 \frac{\xi^i}{i!} \left(\frac{\eta^i}{i!} + (-1)^i \frac{\eta^{5-i}}{(5-i)!} \right), & 0 \leq \eta \leq \xi \leq 1 \\ \sum_{i=0}^2 \frac{\eta^i}{i!} \left(\frac{\xi^i}{i!} + (-1)^i \frac{\xi^{5-i}}{(5-i)!} \right), & 0 \leq \xi \leq \eta \leq 1 \end{cases}$$

or

$$V(\eta, \xi) = \begin{cases} 1 + \eta\xi + \frac{(\eta^5 - 5\eta^4\xi + 10\eta^3\xi^2 + 30\eta^2\xi^3)}{120}, & 0 \leq \eta \leq \xi \leq 1 \\ 1 + \eta\xi + \frac{(\xi^5 - 5\xi^4\eta + 10\xi^3\eta^2 + 30\xi^2\eta^3)}{120}, & 0 \leq \xi \leq \eta \leq 1 \end{cases} \quad (14)$$

Correspondingly

$$\begin{aligned}
\frac{\partial V(\eta, \xi)}{\partial \xi} &= \frac{\eta(4\eta\xi(\eta+3) - \eta^3)}{24} + \eta, \\
\frac{\partial^2 V(\eta, \xi)}{\partial \eta \partial \xi} &= -\frac{\eta^3}{6} + \frac{\eta\xi(\eta+2)}{2} + 1, \\
\frac{\partial^2 V(\eta, \xi)}{\partial \xi^2} &= \frac{\eta^2(\eta+3)}{6}, \\
\frac{\partial^3 V(\eta, \xi)}{\partial \eta \partial \xi^2} &= \frac{\eta^2}{2} + \eta, \\
\frac{\partial^4 V(\eta, \xi)}{\partial \eta^2 \partial \xi^2} &= \eta + 1, \\
0 &\leq \eta \leq \xi \leq 1.
\end{aligned} \tag{15}$$

In addition, the following formulae are required for computing the normal spline derivatives

$$\begin{aligned}
\frac{\partial V(\eta, \xi)}{\partial \eta} &= \frac{\eta^4 - 4\xi(\eta^3 - 6) + 6\eta\xi^2(\eta+2)}{24}, \\
\frac{\partial^2 V(\eta, \xi)}{\partial \eta^2} &= \frac{\eta^3 - 3\eta^2\xi + 3\xi^2(\eta+1)}{6}, \\
\frac{\partial^3 V(\eta, \xi)}{\partial \eta^2 \partial \xi} &= -\frac{\eta^2}{2} + \eta\xi + \xi, \\
0 &\leq \eta \leq \xi \leq 1.
\end{aligned} \tag{16}$$

Thereby we can construct a normal interpolating spline in interval $[0, 1]$. Solving the interpolating problem in an arbitrary interval can be done by mapping the latter to $[0, 1]$ through affine change of variable. Define constants a and b as

$$a = \min(x_1, s_1, t_1), \quad b = \max(x_{n_1}, s_{n_2}, t_{n_3}),$$

and introduce values $\bar{x}_i, \bar{s}_j, \bar{t}_k$:

$$\begin{aligned}
\bar{x}_i &= \frac{x_i - a}{b - a}, \quad \bar{s}_j = \frac{s_j - a}{b - a}, \quad \bar{t}_k = \frac{t_k - a}{b - a}, \\
i &= 1, \dots, n_1, \quad j = 1, \dots, n_2, \quad k = 1, \dots, n_3.
\end{aligned}$$

Then original Problem (1) is transformed to

Problem: Given points $0 \leq \bar{x}_1 < \bar{x}_2 < \dots < \bar{x}_{n_1} \leq 1$, $0 \leq \bar{s}_1 < \bar{s}_2 < \dots < \bar{s}_{n_2} \leq 1$ and $0 \leq \bar{t}_1 < \bar{t}_2 < \dots < \bar{t}_{n_3} \leq 1$ find a function \bar{f} such that

$$\begin{aligned}
\bar{f}(\bar{x}_i) &= u_i, \quad i = 1, 2, \dots, n_1, \\
\bar{f}'(\bar{s}_j) &= v_j(b - a), \quad j = 1, 2, \dots, n_2, \\
\bar{f}''(\bar{t}_k) &= w_k(b - a)^2, \quad k = 1, 2, \dots, n_3.
\end{aligned} \tag{17}$$

Assuming $\bar{\sigma}(\bar{\eta})$ is a normal spline constructed for the Problem (17), the normal spline $\sigma(\eta)$ can be received as

$$\sigma(\eta) = \bar{\sigma}\left(\frac{\eta - a}{b - a}\right), \quad a \leq \eta \leq b.$$

Reproducing kernel for Bessel potential space was presented in [5] and its simplified variant in [14], [3], [18], [19]. Here normal splines will be constructed in Bessel potential space $H_\varepsilon^3(R)$ defined as

$$H_\varepsilon^3(R) = \left\{ f \mid f \in S', (\varepsilon^2 + |s|^2)^{3/2} \mathcal{F}[f] \in L_2(R) \right\}, \varepsilon > 0,$$

where $S'(R)$ is space of L. Schwartz tempered distributions and $\mathcal{F}[f]$ is a Fourier transform of the f ([5], [3], [17]). The parameter ε may be treated as a "scaling parameter". It allows to control approximation properties of the normal spline which usually are getting better with smaller values of ε , also it may be used to reduce the illconditioness of the related computational problem (in traditional theory $\varepsilon = 1$) ([14], [20]). Space $H_\varepsilon^3(R)$ is a Hilbert space with norm

$$\|f\|_{H_\varepsilon^3} = \|(\varepsilon^2 + |s|^2)^{3/2} \mathcal{F}[\varphi]\|_{L_2},$$

it is continuously embedded in the Hölder space $C_b^3(R)$ that consists of all functions having bounded continuous derivatives up to order 2 ([3]). The reproducing kernel of this space is defined up to a constant as follows ([14], [20])

$$V(\eta, \xi, \varepsilon) = \frac{1}{\varepsilon^2} \exp(-\varepsilon|\xi - \eta|)(3 + 3\varepsilon|\xi - \eta| + \varepsilon^2|\xi - \eta|^2). \quad (18)$$

Correspondingly

$$\begin{aligned} \frac{\partial V(\eta, \xi, \varepsilon)}{\partial \xi} &= -\exp(-\varepsilon|\xi - \eta|)(\xi - \eta)(\varepsilon|\xi - \eta| + 1), \\ \frac{\partial^2 V(\eta, \xi, \varepsilon)}{\partial \eta \partial \xi} &= -\exp(-\varepsilon|\xi - \eta|)(\varepsilon|\xi - \eta|(\varepsilon|\xi - \eta| - 1) - 1), \\ \frac{\partial^2 V(\eta, \xi, \varepsilon)}{\partial \xi^2} &= \exp(-\varepsilon|\xi - \eta|)(\varepsilon|\xi - \eta|(\varepsilon|\xi - \eta| - 1) - 1), \\ \frac{\partial^3 V(\eta, \xi, \varepsilon)}{\partial \eta \partial \xi^2} &= \varepsilon^2 \exp(-\varepsilon|\xi - \eta|)(\xi - \eta)(\varepsilon|\xi - \eta| - 3), \\ \frac{\partial^2 V(\eta, \xi, \varepsilon)}{\partial \eta^2 \partial \xi^2} &= \varepsilon^2 \exp(-\varepsilon|\xi - \eta|)(\varepsilon|\xi - \eta|(\varepsilon|\xi - \eta| - 5) + 3). \end{aligned} \quad (19)$$

In addition, the following formulae are required for computing the normal spline derivatives

$$\begin{aligned} \frac{\partial V(\eta, \xi, \varepsilon)}{\partial \eta} &= \exp(-\varepsilon|\xi - \eta|)(\xi - \eta)\varepsilon|\xi - \eta| + 1, \\ \frac{\partial^2 V(\eta, \xi, \varepsilon)}{\partial \eta^2} &= \exp(-\varepsilon|\xi - \eta|)(\varepsilon|\xi - \eta|(\varepsilon|\xi - \eta| - 1) - 1), \\ \frac{\partial^3 V(\eta, \xi, \varepsilon)}{\partial \eta^2 \partial \xi} &= -\varepsilon^2 \exp(-\varepsilon|\xi - \eta|)(\xi - \eta)(\varepsilon|\xi - \eta| - 3). \end{aligned} \quad (20)$$

In a case when there is no information of second derivative values of function f the Problem (1) is reducing to the following one:

Problem: Given points $x_1 < x_2 < \dots < x_{n_1}$, $s_1 < s_2 < \dots < s_{n_2}$ find a function f such that

$$\begin{aligned} f(x_i) &= u_i, \quad i = 1, 2, \dots, n_1, \\ f'(s_j) &= v_j, \quad j = 1, 2, \dots, n_2, \\ n_1 &\geq 0, \quad n_2 \geq 0. \end{aligned} \quad (21)$$

We assume here that f is a continuously differentiable function. It can be treated as an element of Sobolev space $W_2^2[a, b]$ or Bessel potential space $H_\varepsilon^2(R)$. First space is continuously embedded in space $C^1[a, b]$ of functions continuous with their first derivative and the second one is continuously embedded in Hölder space $C_b^1(R)$ of continuously differentiable functions with bounded derivative.

Reproducing kernel of Sobolev spaces $W_2^2[0, 1]$ with norm

$$\|f\|_{W_2^2} = \left(\sum_{i=0}^1 (f^{(i)}(0))^2 + \int_0^1 (f^{(2)}(s))^2 ds \right)^{1/2},$$

was received in [11]:

$$V(\eta, \xi) = \begin{cases} 1 + (\eta + \eta^2/2)\xi - \eta^3/6, & 0 \leq \eta \leq \xi \leq 1 \\ 1 + (\xi + \xi^2/2)\eta - \xi^3/6, & 0 \leq \xi \leq \eta \leq 1 \end{cases} \quad (22)$$

Correspondingly

$$\begin{aligned} \frac{\partial V(\eta, \xi)}{\partial \xi} &= \eta^2/2 + \eta, \\ \frac{\partial^2 V(\eta, \xi)}{\partial \eta \partial \xi} &= \eta + 1, \\ 0 \leq \eta \leq \xi \leq 1. \end{aligned} \quad (23)$$

In addition, the following formula is required for computing the normal spline derivative

$$\begin{aligned} \frac{\partial V(\eta, \xi)}{\partial \eta} &= -\eta^2/2 + \eta\xi + \xi, \\ 0 \leq \eta \leq \xi \leq 1. \end{aligned} \quad (24)$$

Reproducing kernel for Bessel potential space $H_\varepsilon^2(R)$ with norm

$$\|f\|_{H_\varepsilon^2} = \|(\varepsilon^2 + |s|^2)\mathcal{F}[\varphi]\|_{L_2}.$$

is defined by ([14], [20]):

$$V(\eta, \xi, \varepsilon) = \frac{1}{\varepsilon^2} \exp(-\varepsilon|\xi - \eta|)(1 + \varepsilon|\xi - \eta|). \quad (25)$$

Correspondingly

$$\begin{aligned} \frac{\partial V(\eta, \xi, \varepsilon)}{\partial \xi} &= -\exp(-\varepsilon|\xi - \eta|)(\xi - \eta), \\ \frac{\partial^2 V(\eta, \xi, \varepsilon)}{\partial \eta \partial \xi} &= \exp(-\varepsilon|\xi - \eta|)(1 - \varepsilon|\xi - \eta|). \end{aligned} \quad (26)$$

In addition, the following formula is required for computing the normal spline derivative

$$\frac{\partial V(\eta, \xi, \varepsilon)}{\partial \eta} = \exp(-\varepsilon|\xi - \eta|)(\xi - \eta). \quad (27)$$

In a case when there is no information of function f derivatives the Problem (1) is reducing to the simplest interpolation problem:

Problem: Given points $x_1 < x_2 < \dots < x_{n_1}$ find a function f such that

$$\begin{aligned} f(x_i) &= u_i, \quad i = 1, 2, \dots, n_1, \\ n_1 &> 0. \end{aligned} \quad (28)$$

We assume that f is a continuous function. It can be treated as an element of Sobolev space $W_2^1[a, b]$ or Bessel potential space $H_\varepsilon^1(R)$. First space is continuously embedded in space $C[a, b]$ of continuous functions and the second one is continuously embedded in Hölder space $C_b^{1/2}(R)$ of continuous and bounded functions.

Reproducing kernel of Sobolev spaces $W_1^2[0, 1]$ with norm

$$\|f\|_{W_2^1} = \left((f^2(0) + \int_0^1 (f^{(1)}(s))^2 ds) \right)^{1/2},$$

was received in [11]:

$$V(\eta, \xi) = \begin{cases} 1 + \eta, & 0 \leq \eta \leq \xi \leq 1 \\ 1 + \xi, & 0 \leq \xi \leq \eta \leq 1 \end{cases} \quad (29)$$

Reproducing kernel for Bessel potential space $H_\varepsilon^1(R)$ with norm

$$\|f\|_{H_\varepsilon^1} = \|(\varepsilon^2 + |s|^2)^{1/2} \mathcal{F}[\varphi]\|_{L_2}.$$

is defined by ([14], [20]):

$$V(\eta, \xi, \varepsilon) = \exp(-\varepsilon|\xi - \eta|). \quad (30)$$

The normal splines method for one-dimensional function interpolation and linear ordinary differential and integral equations was proposed in [11] and developed in [8], [9], [10]. Multidimensional normal splines method was developed for two-dimensional problem of low-range computerized tomography in [15] and applied for solving a mathematical economics problem in [12]. Further results were reported on seminars and conferences.

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