

# Iterative Learning Control with Nonlinear Measurements

Aaron Trowbridge

## Setup

Given a nominal state trajectory  $\hat{x}(t)$  and control trajectory  $u(t)$ , we apply the controls to the experimental system and retrieve a set of measurements (abusing notation)  $y(t)$  from the (possibly hidden) experimental trajectory  $\bar{x}(t)$ . Schematically we have:

$$\begin{array}{c}
 u(t) \longrightarrow \overbrace{\hat{x}(t) \longrightarrow g(\hat{x}(t)) = \hat{y}(t)}^{\text{model}} \\
 \searrow \hspace{10em} \swarrow \\
 \hspace{10em} \underbrace{\bar{x}(t) \longrightarrow g(\bar{x}(t)) = \bar{y}(t)}_{\text{experiment}} \longrightarrow \Delta y = \bar{y} - \hat{y}
 \end{array}$$

$$u^k \longrightarrow \bar{y}^k \longrightarrow \Delta y^k \longrightarrow \text{QP}(\Delta y^k) \longrightarrow (\Delta x, \Delta u) \longrightarrow u^{k+1}$$

which coincides with the simplified model situation:

$$u(t) \longrightarrow \hat{x}(u(t), t) \longrightarrow g(\hat{x}(t), t) = \hat{y}(t)$$

We now have two sets of measurements:

- $\hat{y}(t)$  : the nominal measurement
- $\bar{y}(t)$  : the experimental measurement

## Problem Formulation

Let us write

$$\bar{x}(t) = \hat{x}(t) + e(t)$$

where  $e(t)$  is the error in the experimental trajectory. To correct for this error, we can find a correction term  $\Delta x(t)$  s.t.

$$g(\bar{x} + \Delta x) = g(\hat{x} + e + \Delta x) = \hat{y}$$

For example, if  $g(x) = x$  is the identity function, i.e. we are trying to track the trajectory:

$$\Delta x = -e$$

The real problem involves finding the corresponding correction to the controls:  $\Delta u(t)$ . This involves setting up a quadratic optimization problem.

## Quadratic Correction Problem

The goal is now to go from the measurement error  $\Delta y$  to a state correction  $\Delta x$  and a control correction  $\Delta u$  by simultaneously solving two linear systems. Schematically:

$$\Delta y \xrightarrow[g]{M \cdot \Delta x = \Delta y} \Delta x \xrightarrow[f]{D \cdot \Delta z = 0} \Delta u$$

### Measurement Correction to State Correction

With  $\Delta y \equiv \bar{y} - \hat{y}$ , we have

$$\begin{aligned} \bar{y} &= g(\bar{x}) \\ &= g(\hat{x} + e) \\ &\approx g(\hat{x}) + \frac{\partial g}{\partial \hat{x}} \cdot e \\ &= \hat{y} + \frac{\partial g}{\partial \hat{x}} \cdot e \end{aligned}$$

which, with writing  $\hat{M} = \partial g / \partial \hat{x}$  yields

$$\Delta y \approx \hat{M} \cdot e$$

and since  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  where  $m \leq n$ ,  $\hat{M} \in \mathbb{R}^{m \times n}$  is not necessarily invertible, but we can use the Moore-Penrose pseudoinverse here to get a guess for  $e$ :

$$\boxed{e \approx \hat{M}^+ \cdot \Delta y \equiv \hat{e}}$$

To tie the experimental measurements to the model measurements we require

$$\begin{aligned} \hat{y} &= g(\bar{x} + \Delta x) \\ &\approx \bar{y} + \frac{\partial g}{\partial \bar{x}} \cdot \Delta x \\ &= \bar{y} + \bar{M} \cdot \Delta x \end{aligned}$$

which yields the condition

$$\boxed{\bar{M} \cdot \Delta x = -\Delta y} \tag{1}$$

where

$$\begin{aligned}
\bar{M}_i^j &= \partial g(\hat{x} + e)_i^j \\
&\approx \partial g(\hat{x} + \hat{e})_i^j \\
&\approx \partial g(\hat{x})_i^j + \sum_k (\partial^2 g(\hat{x}))_i^{jk} \hat{e}_k \\
&= \hat{M}_i^j + \sum_{kl} (\partial^2 g(\hat{x}))_i^{jk} \left( \hat{M}^+ \right)_k^l \Delta y_l
\end{aligned}$$

where

$$\partial g(\cdot) = \left. \frac{\partial g}{\partial x} \right|_{x= \cdot}.$$

### State Correction to Control Correction

To propagate the state correction to the control correction, we utilize the dynamics constraint,  $f(z_t, z_{t+1}) = 0$ , where we define the *knot point*

$$z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}$$

Let's write  $\mathbf{z}_t = \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix}$ . Then we have

$$\begin{aligned}
0 &= f(\hat{\mathbf{z}}_t + \Delta \mathbf{z}_t) \\
&\approx f(\hat{\mathbf{z}}_t) + \partial f(\hat{\mathbf{z}}_t) \cdot \Delta \mathbf{z}_t
\end{aligned}$$

which yields, with  $\hat{D} = \partial f(\hat{\mathbf{z}}_t)$

$$\boxed{\hat{D} \cdot \Delta \mathbf{z}_t = 0} \tag{2}$$

### Putting it all together

We seek to find the solution to

$$\begin{aligned}
&\underset{\Delta x_{1:T}, \Delta u_{1:T}}{\text{minimize}} && \frac{1}{2} \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\
&\text{subject to} && \bar{M} \cdot \Delta x_\tau = -\Delta y_\tau \quad \forall \tau \\
&&& \hat{D} \cdot \Delta \mathbf{z}_t = 0 \quad \forall t
\end{aligned}$$

where the  $\tau$ s are the measurement times.

Building the KKT matrix from this problem, we can solve the system and extract  $\Delta u(t)$  and repeat the procedure until convergence.

This problem, which returns  $\Delta Z$  is referred to as

$$\Delta Z = \text{QuadraticProblem}(\hat{Z}, \Delta Y)$$

### KKT Matrix (for just single quantum state and controls)

Below we use:

- $n = \dim z_t = \dim x_t + \dim u_t$
- $d = \dim x_t = \dim f(z_t, z_{t+1})$
- $c = \dim u_t$
- $m = \dim y_t$
- $M = \#$  of measurements

For a trajectory  $Z = \text{vec}(z_{1:T})$ , we need to construct the matrix

$$\begin{pmatrix} H & A^\top \\ A & 0 \end{pmatrix}$$

where  $H$  is the Hessian of the cost function:

$$H = \bigoplus_{t=1}^T (Q \oplus R) = I^{T \times T} \otimes (Q \oplus R)$$

and  $A$  is the constraint Jacobian:

$$A = \begin{pmatrix} \partial F \\ \partial G \end{pmatrix}$$

with

$$\partial F = \begin{pmatrix} \partial f(\hat{\mathbf{z}}_1) & & \\ & \ddots & \\ & & \partial f(\hat{\mathbf{z}}_{T-1}) \end{pmatrix} \in \mathbb{R}^{d(T-1) \times nT}$$

and

$$\partial G = \begin{pmatrix} \ddots & & \\ & \partial g(\bar{x}_\tau) \mathbf{0}^{m \times (a+c)} & \\ & & \ddots \end{pmatrix} \in \mathbb{R}^{mM \times nT}$$

where  $\tau = t_1, \dots, t_M$  are the measurement times.

For the constraints we then have

$$\partial F \cdot \Delta Z = 0 \quad \text{and} \quad \partial G \cdot \Delta Z = -\Delta Y$$

where again

$$\Delta Y = \bar{Y} - \hat{Y}$$

## An Alternative Quadratic Problem

In the regime of noisy measurements, satisfying both the dynamics constraints and the measurement constraints becomes infeasible. To overcome this we can relax the measurement into a maximum likelihood problem by assuming additive gaussian noise  $w \sim \mathcal{N}(0, \Sigma)$ . To see this let's write

$$\begin{aligned} \bar{y} = \hat{y} - M \cdot \Delta x + w &\implies \bar{y} \sim \mathcal{N}(\hat{y} - M \cdot \Delta x, \Sigma) \\ &\implies \Delta y \sim \mathcal{N}(-M \cdot \Delta x, \Sigma) \end{aligned}$$

where  $\Sigma$  is the covariance matrix of the measurement noise, which we can get from the experiment. To make the following clearer, let's define the parameterized distribution over  $\Delta y$  s.t.

$$\Delta y \sim p(\Delta x) = \mathcal{N}(-M \cdot \Delta x, \Sigma)$$

then, given an observation  $\bar{y}$ , we can find the MLE for the parameter  $\Delta x$  as the solution to the following optimization problem:

$$\begin{aligned} \max_{\Delta x} \log p(\Delta x) &\implies \min_{\Delta x} \frac{1}{2} (\Delta y + M \cdot \Delta x)^\top \Sigma^{-1} (\Delta y + M \cdot \Delta x) \\ &\implies \min_{\Delta x} \frac{1}{2} \Delta x^\top (M^\top \Sigma^{-1} M) \Delta x + (\Delta y^\top \Sigma^{-1} M) \Delta x \end{aligned}$$

We can then augment our initial problem with this objective term and remove the measurement constraint. This yields the following problem:

$$\begin{aligned} &\underset{\Delta x_{1:T}, \Delta u_{1:T}}{\text{minimize}} && \frac{1}{2} \sum_t \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \\ &&& + \sum_\tau \frac{1}{2} \Delta x_\tau^\top (M_\tau^\top \Sigma^{-1} M_\tau) \Delta x_\tau + (\Delta y_\tau^\top \Sigma^{-1} M_\tau) \Delta x_\tau \\ &\text{subject to} && \hat{D} \cdot \Delta \mathbf{z}_t = 0 \quad \forall t \\ &&& -u_{\max} - \hat{u}_t < \Delta u_t < u_{\max} - \hat{u}_t \\ &&& \Delta u_1 = \Delta u_T = 0 \end{aligned}$$

## ILC Algorithm

Tying everything together, *iterative learning control* (ILC) solves the aforementioned quadratic problem and updates the trajectory iteratively until convergence. The following algorithm codifies this:

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### Algorithm 1: Iterative Control Learning

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**Data:**  $\hat{Z}^{\text{goal}}$ ,  $\text{tol} > 0$ ,  $\alpha = 0.5$ ,  $\beta = 0.1$ ,  
**Result:**  $U$   
 $Y^{\text{goal}} \leftarrow \text{measure}(\hat{Z}^{\text{goal}}) = \text{vec}(y_{\tau_1:\tau_M})$   
 $\hat{Z} \leftarrow \hat{Z}^{\text{goal}}$   
 $U \leftarrow \text{controls}(\hat{Z}) = \text{vec}(u_{1:T})$   
 $\bar{Y} \leftarrow \text{experiment}(U) = \text{vec}(\bar{y}_{\tau_1:\tau_M})$   
 $\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$   
 $k \leftarrow 1$   
**while**  $|\Delta Y| > \text{tol}$  **do**  
     $\Delta Z \leftarrow \beta \cdot \text{QuadraticProblem}(\hat{Z}, \Delta Y)$   
     $\hat{Z}_{\text{next}} \leftarrow \hat{Z} + \Delta Z$   
     $\bar{y}_{T,\text{next}} \leftarrow \text{measure\_final\_state}(\hat{Z}_{\text{next}})$   
     $\Delta y_{T,\text{next}} \leftarrow \bar{y}_{T,\text{next}} - \bar{y}_{T,\text{goal}}$   
    **while**  $\|\Delta y_{T,\text{next}}\|_p > \|\Delta y_T\|$ ;      // Backtracking line search  
        **do**  
             $\Delta Z \leftarrow \alpha \cdot \Delta Z$   
             $\hat{Z}_{\text{next}} \leftarrow \hat{Z} + \Delta Z$   
             $\bar{y}_{T,\text{next}} \leftarrow \text{measure\_final\_state}(\hat{Z}_{\text{next}})$   
             $\Delta y_{T,\text{next}} \leftarrow \bar{y}_{T,\text{next}} - \bar{y}_{T,\text{goal}}$   
        **end**  
     $\hat{Z} \leftarrow \hat{Z}_{\text{next}}$   
     $U \leftarrow \text{controls}(\hat{Z})$   
     $\bar{Y} \leftarrow \text{experiment}(U)$   
     $\Delta Y \leftarrow \bar{Y} - Y^{\text{goal}}$   
     $k \leftarrow k + 1$   
**end**  
**return**  $U$

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## 1 Misc.

$$\begin{aligned} & \underset{\Delta x, \Delta u}{\text{minimize}} && \|\Delta y\|^2 = \|\partial_x g \cdot \Delta x\|^2 + \Delta y^\top \cdot \partial_x g \cdot \Delta x \\ & \text{subject to} && \partial_{\mathbf{z}} f \cdot \Delta \mathbf{z} = 0 \end{aligned}$$

$$\begin{aligned} \Delta y &= g(\hat{x} + e + \Delta x) - \hat{y} \\ &\approx g(\hat{x}) + \partial_x g \cdot e + \partial_x g \cdot \Delta x - \hat{y} \end{aligned}$$

$$\begin{aligned} \underset{\Delta x, \Delta u}{\text{minimize}} \quad & \frac{1}{2} \|\Delta y^{k+1}\|^2 = \frac{1}{2} \|g(\bar{x}^k + \Delta x) - \hat{y}\|^2 \\ & \approx \frac{1}{2} \|g(\bar{x}^k) + M\Delta x - \hat{y}\|^2 \\ & = \frac{1}{2} \|\Delta y^k + M\Delta x\|^2 \end{aligned}$$

$$\begin{aligned} & \underset{\Delta x, \Delta u}{\text{minimize}} && \frac{1}{2} \Delta x^\top M^\top M \Delta x + \Delta y^\top M \Delta x \\ & \text{subject to} && \partial f \cdot (\Delta x_{t+1}, \Delta x_t, \Delta u_t)^\top = 0 \end{aligned}$$