The channel-irreducible vertex of the Hubbard atom away from half filling: derivation

Samuel Badr

TU Wien, October 2026

Licensed under CC BY 4.0.

1 Overview

The irreducible vertex Γ_r of the Hubbard atom is known in closed form at half filling [1]. Here we derive it at arbitrary filling (zero magnetic field) in all four channels $r \in \{\mathrm{d}, \mathrm{m}, \mathrm{s}, \mathrm{t}\}$. Appendices F and G also state the vertices and analyze their divergences with simultaneous doping and Zeeman field, in the coupled longitudinal, transverse-spin, opposite-spin pair and equal-spin pair sectors. By definition

$$\Gamma_r = \beta^2\big(\chi_r^{-1} - \eta_r \chi_{0,r}^{-1}\big), \tag{1}$$

where χ_r is the generalized susceptibility, an infinite matrix in two fermionic Matsubara frequencies, and $\chi_{0,r}$ is its bare part. So the task is to invert χ_r in closed form.

At half filling χ_r is a diagonal matrix plus a mirror term plus a matrix of rank at most two, and the inversion is elementary. Away from half filling the mirror term becomes a dense *Cauchy matrix* $C_{nm} = \rho/(z_n + z_m)$ (Figure 1). The particle–hole susceptibility is then no longer a finite-rank perturbation of an easily invertible matrix. Two properties of C still allow a closed-form inverse:

1. Like the mirror it replaces, C is a scaled involution: $C^2 = h^2 \mathbb{1}$ for a number h.
2. C maps a simple pole to a simple pole, with a thermal factor α:

$$\sum_m \frac{C_{nm}}{z_m - x} = \frac{\alpha(x)}{z_n + x}, \qquad \alpha(x)\alpha(-x) = h^2. \tag{2}$$

After multiplication by a polynomial, χ_r is a diagonal matrix plus λC plus a matrix of rank two. Each column of χ_r^{-1} is then built from a diagonal entry, one Cauchy column, a constant, and two pole pairs with one thermal denominator each. All coefficients follow from 2×2 linear systems whose determinants are given by a single quartic.

Results.

- **Triplet:** Γ_t is a diagonal-minus-mirror term (Theorem 2).
- **Singlet:** Γ_s is a diagonal-plus-mirror term, the constant $2U$, and one separable term with a single thermal denominator (Theorem 3).
- **Density and magnetic:** $\Gamma_\mathrm{d}, \Gamma_\mathrm{m}$ have the same anatomy: a diagonal term, the constant $\pm U$, a dense term without thermal factors, and two separable terms with one thermal denominator each (Theorem 4). At half filling the dense term becomes a mirror term and one separable term remains (Corollary 5).
- **Removable singularities:** where a denominator of these formulas vanishes without Γ_r diverging, the limits are given explicitly (Proposition 6).

The inverses are two-sided and agree with the limit of inverses on finite frequency boxes, i.e. with what numerical inversion converges to (Proposition 7).

Roadmap.

- Section 2 defines all quantities; Section 3 states the input, the susceptibility from the Lehmann representation.

- Section 4 states all results. The rest of the note proves them.
- Section 5 evaluates the Matsubara sums; Section 6 proves the particle–particle results, which need no Cauchy matrix.
- Section 7 develops the algebra of C, which Section 8 uses for the particle–hole channels. The limit $\mu \to 0$, the removable singularities and the finite-box statement are proved in Appendix B.
- Appendix F writes the vertices out in terms of U, β, μ and the frequencies, for implementation; Appendix G derives and plots the divergences of Γ_r; Appendix H lists the symbols with the place where each is defined.

What is assumed. The zero-field results, except the numerical parts of Appendix G, are proved from the input Eqs. 18–20 (Ref. [2]) and the rational identities of Appendix A (checked by computer algebra). The Mathematica notebook `mathematica/doped_gamma.nb` derives the input from the Hamiltonian and replays the proofs step by step (Appendix D). Independent numerical checks are listed in Appendix D. Most of the proofs are also formalized in Lean 4, which also derives the input from the Hamiltonian; statements marked Lean or Lean, in part are formalized, and the table in Appendix E says how much of each. The field formulas are stated without new derivations; their domains and the numerical status of longitudinal axis and exceptional-set extensions are specified in Appendix F.6.5. They are not covered by the Lean formalization. Numerical stability in floating point is not discussed here.

Implementation. The zero-field results are implemented as `DopedHubbardAtom` in the Julia package HubbardAtoms.jl, which also evaluates them stably at low temperatures and near the removable singularities, and is tested against exact diagonalization and finite-box inversion. `FieldHubbardAtom` implements the four field sectors of Appendix F.6.

2 Setup

2.1 Model and thermal weights

The atom is

$$\mathcal{H} = U\Big(n_\uparrow - \frac{1}{2}\Big)\Big(n_\downarrow - \frac{1}{2}\Big) - \mu\big(n_\uparrow + n_\downarrow - 1\big), \quad a := \frac{U}{2}, \tag{3}$$

with real interaction U, chemical potential μ (measured from half filling, $\mu = 0$) and inverse temperature $\beta > 0$. Its four states have

state	$\lvert 0\rangle$	$\lvert\uparrow\rangle$	$\lvert\downarrow\rangle$	$\lvert\uparrow\downarrow\rangle$
energy	$\frac{U}{4} + \mu$	$-\frac{U}{4}$	$-\frac{U}{4}$	$\frac{U}{4} - \mu$
Boltzmann weight	p_0	p_1	p_1	p_2

Here p_1 is the weight of *each* singly occupied state, so

$$p_0 + 2p_1 + p_2 = 1, \qquad \frac{p_2}{p_0} = e^{2\beta\mu}, \qquad \frac{p_1}{p_0} = e^{\beta(a+\mu)}, \qquad \frac{p_1}{p_2} = e^{\beta(a-\mu)}. \tag{4}$$

Three combinations of the weights recur:

$$\begin{aligned} &\rho := p_2 - p_0 = \langle n_\uparrow + n_\downarrow\rangle - 1, && \text{the filling; } \rho = 0 \text{ at half filling} \\ &h := \beta\sqrt{p_0 p_2}, && C^2 = h^2\mathbb{1} \text{ (Lemma 13)} \\ &\overline{c} := \beta(p_0 + p_2)/2. && \mathrm{PV}\sum_m C_{nm} = \overline{c} \text{ (Lemma 15)} \end{aligned} \tag{5}$$

By Eq. 4,

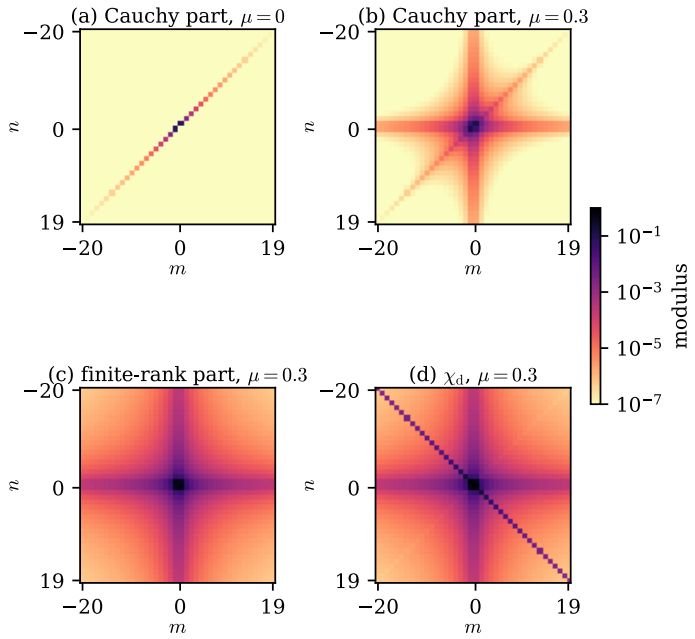


Figure 1: Structure of χ_d at $U = 2$, $\beta = 3$, $\omega = 0$, as moduli of matrix elements in the fermionic indices n, m. (a) At half filling the pair term is a mirror, nonzero only for $m = -n - 1$. (b) Away from half filling it becomes the dense Cauchy matrix C. (c) The remaining finite-rank part. (d) The full χ_d; the diagonal dominates.

$$p_1 + p_2 = \frac{1+\rho}{2}, \qquad p_0 + p_1 = \frac{1-\rho}{2}, \qquad \rho^2 + 4p_0p_2 = (p_0+p_2)^2, \qquad \bar{c}^2 - h^2 = \beta^2\rho^2/4, \tag{6}$$

and for $\mu \neq 0$, inserting $p_2 = p_0 e^{2\beta\mu}$,

$$\begin{aligned} &\rho = 2p_0 e^{\beta\mu}\sinh(\beta\mu), \quad p_0 + p_2 = 2p_0 e^{\beta\mu}\cosh(\beta\mu) \\ \Longrightarrow \quad &\rho\coth(\beta\mu) = p_0 + p_2, \quad \rho^2 = 4p_0p_2\sinh^2(\beta\mu). \end{aligned} \tag{7}$$

2.2 Frequencies and propagator

The fermionic and bosonic Matsubara frequencies are

$$\nu_n = (2n+1)\frac{\pi}{\beta}, \qquad \Omega_\ell = \frac{2\ell\pi}{\beta} \qquad (n, \ell \in \mathbb{Z}). \tag{8}$$

The bosonic transfer $\omega_k := \Omega_k$ is fixed throughout. Write

$$x_n := \mu + i\nu_n, \qquad w := i\omega_k, \qquad J(x) := \frac{1}{x^2 - a^2}. \tag{9}$$

The propagator

$$G(\nu) := -\int_0^\beta \mathrm{d}\tau\, e^{i\nu\tau}\langle T c_\uparrow(\tau)c_\uparrow^\dagger\rangle \tag{10}$$

follows from the Lehmann representation: $c_\uparrow^\dagger$ maps $|0\rangle \to |\uparrow\rangle$ (energy change $-a - \mu$, weight $p_0 + p_1$) and $|\downarrow\rangle \to |\uparrow\downarrow\rangle$ (energy change $a - \mu$, weight $p_1 + p_2$). So

$$G(\nu_n) = \frac{p_0+p_1}{x_n + a} + \frac{p_1+p_2}{x_n - a} = G(x_n) := \frac{x_n + a\rho}{x_n^2 - a^2}, \tag{11}$$

where we use the same letter for the rational function $G(x)$.

2.3 Susceptibilities and the irreducible vertex

Following Ref. [1], the generalized susceptibility is

$$\begin{aligned}\chi_{\sigma\sigma'}(\nu,\nu',\omega) \coloneqq \iiint_0^\beta \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\, e^{-i\nu\tau_1+i(\nu+\omega)\tau_2-i(\nu'+\omega)\tau_3} \\ \times\Big[\langle Tc_\sigma^\dagger(\tau_1)c_\sigma(\tau_2)c_{\sigma'}^\dagger(\tau_3)c_{\sigma'}(0)\rangle - \langle Tc_\sigma^\dagger(\tau_1)c_\sigma(\tau_2)\rangle\langle Tc_{\sigma'}^\dagger(\tau_3)c_{\sigma'}(0)\rangle\Big],\end{aligned} \tag{12}$$

and its particle–particle form is

$$\chi_{\mathrm{pp},\sigma\sigma'}(\nu,\nu',\omega) \coloneqq \chi_{\sigma\sigma'}(\nu,\nu',-\omega-\nu-\nu'). \tag{13}$$

The channels and bare susceptibilities are

$$\chi_\mathrm{d} = \chi_{\uparrow\uparrow}+\chi_{\uparrow\downarrow},\quad \chi_\mathrm{m} = \chi_{\uparrow\uparrow}-\chi_{\uparrow\downarrow},\quad \chi_\mathrm{s} = \frac{-\chi_{\mathrm{pp},\uparrow\uparrow}+2\chi_{\mathrm{pp},\uparrow\downarrow}-2\chi_{0,\,\mathrm{pp}}}{4},\quad \chi_\mathrm{t} = \frac{\chi_{\mathrm{pp},\uparrow\uparrow}+2\chi_{0,\,\mathrm{pp}}}{4}, \tag{14}$$

$$\chi_{0,r}(\nu,\nu',\omega) = \delta_{\nu\nu'}B(\nu),\qquad B(\nu) = \begin{cases} -\beta G(\nu)G(\nu+\omega) & (r=\mathrm{d},\mathrm{m}),\\ -\frac{\beta}{2}\,G(\nu)G(-\nu-\omega) & (r=\mathrm{s},\mathrm{t}),\end{cases} \tag{15}$$

with $\chi_{0,\,\mathrm{pp}} \coloneqq \chi_{0,\,\mathrm{s}}$. At fixed ω_k all of these are matrices indexed by (n,m) with $\nu=\nu_n$, $\nu'=\nu_m$; we write $B_n \coloneqq B(\nu_n)$ and $\boldsymbol{B} \coloneqq \mathrm{diag}(B_n)$. The channel-irreducible vertex is

$$\Gamma_r = \beta^2(\chi_r^{-1} - \eta_r\boldsymbol{B}^{-1}),\qquad \eta_\mathrm{d}=\eta_\mathrm{m}=\eta_\mathrm{t}=1,\quad \eta_\mathrm{s}=-1. \tag{16}$$

2.4 Lattice and mirror

Define the centered lattice and the mirror index

$$\begin{gathered} z_n \coloneqq x_n + \frac{w}{2} = \mu + i\pi\frac{2n+1+k}{\beta},\qquad \overline{n} \coloneqq -n-k-1,\\ z_{\overline{n}} = 2\mu - z_n,\qquad x_{\overline{n}} = 2\mu - w - x_n.\end{gathered} \tag{17}$$

The lattice lies on the line $\mathrm{Re}\, z=\mu$, and the mirror reflects it about the real axis (Figure 2).

Conventions.

- The mirror matrix is $\mathcal{M}_{nm} \coloneqq \delta_{m\overline{n}}$, with $\mathcal{M}^2=\mathbb{1}$. A *mirror fixed point* is an n with $n=\overline{n}$; it exists only for odd k.
- Products of infinite matrices are formed entry by entry, $(XY)_{nm}=\sum_l X_{nl}Y_{lm}$.
- A sum $\sum_n$ is the limit of partial sums over the mirror-symmetric boxes $|2n+1+k|\le 2M+1$, $M\to\infty$. We write PV ("principal value") where it converges only in this sense.
- Transposes are ordinary, not Hermitian.
- A function φ on $\mathbb{C}$ is *proper rational* if it is rational and $\varphi(t)\to 0$ for $t\to\infty$.
- Symbols introduced inside a proof are local to it; all others are listed in Appendix H.

3 Input: the generalized susceptibility

The Lehmann representation of $\chi_{\sigma\sigma'}$ for the atom is evaluated in Ref. [2] [Eq. (D3)]. At zero field and in our notation it reads

$$\chi_{\uparrow\uparrow,nm} = -\beta G(x_n)G(x_n+w)\delta_{nm} + \beta a^2(1-\rho^2)J(x_n+w)J(x_m)(\delta_{k0}-\delta_{nm}), \tag{18}$$

$$\chi_{\uparrow\downarrow,nm} = -4\beta a^2 p_1 J(x_n)J(x_n+w)\delta_{nm} + 4\beta a^2(p_0p_2-p_1^2)J(x_n)J(x_m)\delta_{k0} + \mathcal{N}_{nm}, \tag{19}$$

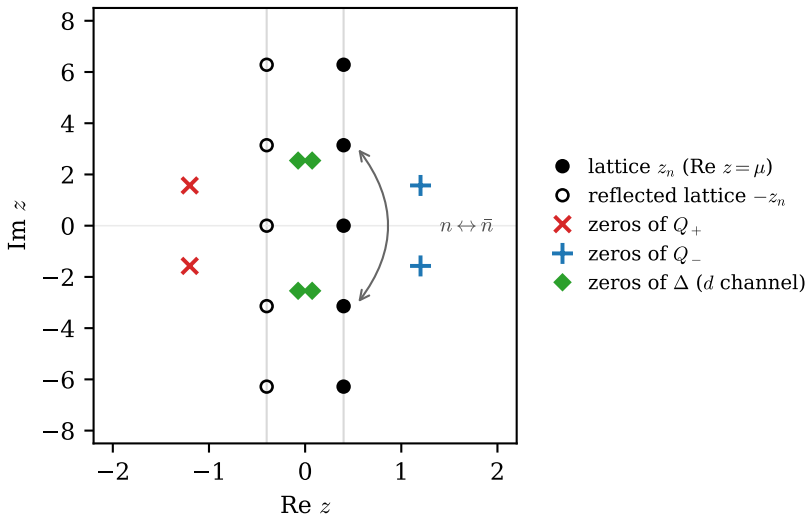


Figure 2: The complex plane for $U = 2.4$, $\beta = 2$, $\mu = 0.4$, $\omega = \omega_1$. The lattice z_n (filled) lies on Re $z = \mu$; the reflected lattice $-z_n$ (open), where the Cauchy kernel $1/(z + z_m)$ is singular, lies on Re $z = -\mu$. Crosses: zeros of $Q_\pm$ (Section 3); diamonds: zeros of e (Section 4.2). None lies on the lattice, so all lattice sums below reduce to finitely many residues.

$$\mathcal{N}_{nm} := c_{n+m+k+1}\Pi(x_n, x_m) + R_{\text{reg}}(x_n, x_m), \qquad c_\ell := \frac{\rho}{2\mu + i\Omega_\ell},$$
$$\Pi(x, y) := \left(\frac{1}{x + w - a} + \frac{1}{y - a}\right)\left(\frac{1}{y + w - a} + \frac{1}{x - a}\right). \tag{20}$$

The pair coefficient c_ℓ stems from the transition $|0\rangle \leftrightarrow |\uparrow\downarrow\rangle$. At $\mu = 0$ it is understood as its limit, $c_\ell = \beta p_0 \delta_{\ell 0}$ (by Eq. 7, $\rho/2\mu \to \beta p_0$). The regular part R_{reg} is a sum of twelve simple fractions, reproduced in Appendix C. We take Eqs. 18–20 as given; they follow exactly from the Lehmann representation (Appendix D), and Lean derives them, with Eq. 11, from Eq. 3 (Appendix E).

Compact form. Define the **Cauchy matrix** $C_{nm} := c_{n+m+k+1}$. Since $2\mu + i\Omega_{n+m+k+1} = z_n + z_m$ by Eq. 17,

$$C_{nm} = \frac{\rho}{z_n + z_m} \quad (\mu \neq 0), \qquad C_{nm} = \beta p_0 \delta_{m\bar{n}} \quad (\mu = 0), \tag{21}$$

and $z_n + z_m = 2\mu + i(\cdots) \neq 0$ for $\mu \neq 0$. Further let

$$Q_\pm(z) := (z \pm a)^2 - \frac{w^2}{4}, \qquad q_0 := a^2 - \frac{w^2}{4} = (U^2 + \omega_k^2)/4. \tag{22}$$

Since $x_n = z_n - w/2$,

$$(x_n - a)(x_n + w - a) = Q_-(z_n), \qquad (x_n + a)(x_n + w + a) = Q_+(z_n), \qquad Q_\pm(-z) = Q_\mp(z). \tag{23}$$

Neither $Q_\pm(z_n)$ vanishes: its factors have imaginary parts ν_n and $\nu_n + \omega_k$, both nonzero.

Lemma 1 (compact input) Lean, in part. With $z = z_n$ and $v = z_m$,

$$\mathcal{N}_{nm} = \frac{4a^2 C_{nm} - 2a}{Q_-(z)Q_-(v)} + \frac{4a^2(1 - \rho)(z + v + 2a)(zv + q_0)}{Q_+(z)Q_-(z)Q_+(v)Q_-(v)}. \tag{24}$$

Proof. Put $y := z + v = x_n + x_m + w$. The two factors of Π combine to

$$\Pi(x_n, x_m) = \frac{y - 2a}{(x_n + w - a)(x_m - a)} \cdot \frac{y - 2a}{(x_m + w - a)(x_n - a)} = \frac{(y - 2a)^2}{Q_-(z)Q_-(v)} \tag{25}$$

by Eq. 23. For $\mu \neq 0$, $C_{nm} = \rho/y$ and $(y - 2a)^2/y = y - 4a + 4a^2/y$, so

$$C_{nm}(y - 2a)^2 = \rho(y - 4a) + 4a^2 C_{nm}. \tag{26}$$

For $\mu = 0$ this holds too, because $\rho = 0$ and $C_{nm} \neq 0$ only where $y = 0$. Thus Eq. 24 is equivalent to

$$R_{\mathrm{reg}}\Big(z - \frac{w}{2}, v - \frac{w}{2}\Big) = \frac{-2a - \rho(z + v - 4a)}{Q_-(z)Q_-(v)} + \frac{4a^2(1 - \rho)(z + v + 2a)(zv + q_0)}{Q_+(z)Q_-(z)Q_+(v)Q_-(v)}, \tag{27}$$

which is the rational identity (A1) of Appendix A. □

4 Results

All lattice sums that occur below are expressed through

$$S_k(x) := \begin{cases} -\frac{\beta}{2}\tanh(\beta(x - \mu)/2) & k \text{ even} \\ -\frac{\beta}{2}\coth(\beta(x - \mu)/2) & k \text{ odd} \end{cases}, \tag{28}$$

which equals $\mathrm{PV}\sum_n (z_n - x)^{-1}$ for x off the lattice (Lemma 8). Appendix F restates the regular vertex formulas with their coefficients written out.

4.1 Singlet and triplet

The particle–particle channels depend on the fermionic index n only through

$$b := 2\mu - w, \qquad t_n := x_n(b - x_n), \tag{29}$$

and $t_{\overline{n}} = t_n$, because $x_{\overline{n}} = b - x_n$ (Eq. 17). Define

$$\begin{aligned} \mathcal{P}(t) &:= \left(t + a^2\right)^2 - a^2 b^2, \\ \mathcal{G}(t) &:= t + a\rho b + a^2\rho^2, \\ A_{\mathrm{s}} &:= a\rho b + a^2(1 - 8p_1), \\ \mathcal{T}_k(t) &:= -\frac{S_k(\mu + \xi)}{\xi}, \quad \xi := \sqrt{b^2/4 - t}. \end{aligned} \tag{30}$$

In terms of these, the bubble is $B_n = -\beta\mathcal{G}(t_n)/(2\mathcal{P}(t_n))$ (Eq. 64), and $\mathcal{T}_k$ is the particle–particle sum (Lemma 10),

$$\mathcal{T}_k(t) = \sum_n \frac{1}{t_n - t}. \tag{31}$$

It is even in ξ, so either branch of the square root may be used; at $\xi = 0$ (even k) it takes the limit value $\beta^2/4$.

> **Theorem 2** (triplet) Lean. Assume $t_n + a\rho b + a^2 \neq 0$ for all $n \neq \overline{n}$. Then for $n \neq \overline{n}$
>
> $$\Gamma_{\mathrm{t},nm} = \frac{\beta a^2(1 - \rho^2)\mathcal{P}(t_n)}{\mathcal{G}(t_n)(t_n + a\rho b + a^2)}(\delta_{nm} - \delta_{m\overline{n}}). \tag{32}$$
>
> At a mirror fixed point $n = \overline{n}$, $\Gamma_{\mathrm{t},nm} = \Gamma_{\mathrm{t},mn} = 0$ for all m.

Theorem 3 (singlet) Lean. Assume $t_n + A_{\mathrm{s}} \neq 0$ for all n and

$$K_{\mathrm{s}} := 1 + \frac{2a}{\beta}\mathcal{T}_k(-A_{\mathrm{s}}) \neq 0. \tag{33}$$

Then

$$\Gamma_{\mathrm{s},nm} = \frac{\beta a^2(\rho^2 - 1 + 8p_1)\mathcal{P}(t_n)}{\mathcal{G}(t_n)(t_n + A_{\mathrm{s}})}(\delta_{nm} + \delta_{m\overline{n}}) + 4a - \frac{4a\,\mathcal{P}(-A_{\mathrm{s}})}{K_{\mathrm{s}}(t_n + A_{\mathrm{s}})(t_m + A_{\mathrm{s}})}. \tag{34}$$

At $\mu = 0$ both formulas reduce to Eq. 19 of Ref. [1] (Appendix D). If $t_c + A_{\mathrm{s}} = 0$ for some c, Eq. 34 has a removable singularity (Proposition 6).

4.2 Density and magnetic

Let $\varepsilon_{\mathrm{d}} := +1$ and $\varepsilon_{\mathrm{m}} := -1$. The particle–hole channels are built from the polynomials

$$\begin{aligned} H(z) &:= z^2 + 2a\rho z + q_1, \quad q_1 := q_0 + 4\varepsilon_r a^2 p_1, \quad \lambda := 4\varepsilon_r a^2, \\ P(z) &:= Q_+(z)Q_-(z), \quad G_2(z) := (z + a\rho)^2 - \frac{w^2}{4}, \end{aligned} \tag{35}$$

and

$$e(z) := H(z)H(-z) - \lambda^2 h^2/\beta^2 = \left(z^2 + q_1\right)^2 - 4a^2\rho^2 z^2 - 16a^4 p_0 p_2 = (z^2 - s_1)(z^2 - s_2). \tag{36}$$

Here $P(z_n)$ times row n of χ_r is $-\beta H(z_n)\delta_{nm} + \lambda C_{nm}$ plus a finite-rank term (Proposition 16); the bubble is $B_n = -\beta G_2(z_n)/P(z_n)$; and $\beta^2 e$ is the determinant of the 2×2 systems in the proof. The roots s_1, s_2 of e in the variable $s = z^2$ solve

$$s^2 + 2(q_1 - 2a^2\rho^2)s + q_1^2 - 16a^4 p_0 p_2 = 0, \tag{37}$$

and u_j denotes either square root of s_j.

The atomic pair. For $r = \mathrm{m}$ and $k = 0$, $q_1 = a^2(1 - 4p_1)$, so $H(a) = 4a^2 p_2$ and $H(-a) = 4a^2 p_0$ by Eq. 4. Then $e(a) = 0$: one root is $s_1 = a^2$, at the double zeros $\pm a$ of P. We call this root pair *atomic*.

The temperature enters only through the thermal function

$$\alpha_k(x) := \rho\, S_k(x) - \overline{c}, \tag{38}$$

and the vertex is assembled from

$$\begin{aligned} F(z) &:= \frac{P(z)}{e(z)}, \quad g_j(z) := \frac{1}{z^2 - s_j}, \\ \mathbb{D}(z, v) &:= \frac{zF(z) - vF(v)}{z^2 - v^2}, \quad \mathbb{D}(z, z) := \frac{F(z) + zF'(z)}{2z}, \\ K_j &:= \beta H(-u_j) + \lambda\alpha_k(u_j), \quad Z_j := \frac{u_j P(u_j)\left[\beta H(-u_j) - \lambda\alpha_k(u_j)\right]}{K_j}, \\ \gamma_j &:= \frac{\lambda\rho Z_j}{s_j - s_\ell} \quad (\ell \neq j). \end{aligned} \tag{39}$$

$\mathbb{D}$ gives the dense term; K_j is the thermal denominator and γ_j the weight of the separable term j.

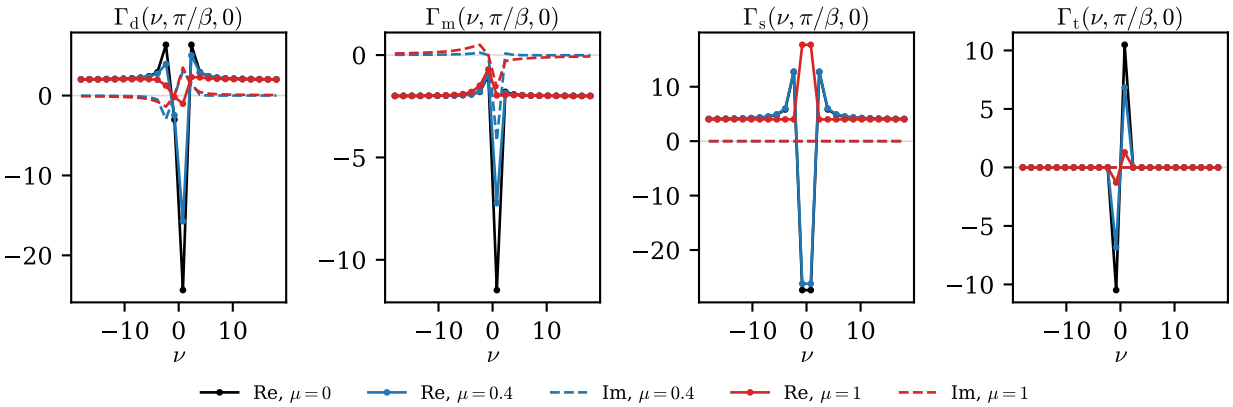


Figure 3: $\Gamma_r(\nu,\pi/\beta,0)$ from Eq. 40 (Eq. 44 at $\mu=0$), Eq. 34, Eq. 32 at $U=2$, $\beta=4$, for $\mu=0$ (half filling, real), $\mu=0.4$ and $\mu=1$. Away from half filling $\Gamma_{\rm d}$, $\Gamma_{\rm m}$ acquire imaginary parts; all Γ_r approach their bare values U, $-U$, $2U$, 0 at large $|\nu|$.

Theorem 4 (density and magnetic) Lean. Let $\mu\neq 0$, $e(z_n)\neq 0$ for all n, $s_1\neq s_2$, and $K_j\neq 0$. Then

$$\Gamma_{r,nm}=\beta P(z_n)\left[\frac{1}{G_2(z_n)}-\frac{H(-z_n)}{e(z_n)}\right]\delta_{nm}+2\varepsilon_r a-\lambda\rho\mathbb{D}(z_n,z_m)+\sum_{j=1}^{2}\gamma_j\, g_j(z_n)g_j(z_m). \tag{40}$$

For $r=\mathrm{m}$ and $k=0$ the atomic pair is omitted, both from the sum and from the assumption $K_j\neq 0$. The formula does not depend on the sign chosen for u_j, and the inverse χ_r^{-1} it represents is two-sided.

So $\Gamma_{\rm d}$ and $\Gamma_{\rm m}$ have the same anatomy as $\Gamma_{\rm s}$: a diagonal term, the constant $\pm U$, a dense term without thermal factors, and two separable terms with one thermal denominator K_j each. Only $\alpha_k(u_j)$, i.e. tanh or coth at the two roots, has to be evaluated. At $s_1=s_2$ and at lattice zeros of e the formula has removable singularities (Proposition 6).

Half filling. At $\mu=0$, $\rho=0$ and Theorem 4 does not apply literally. Here the Cauchy matrix is the mirror, $C=h\mathcal{M}$ with $h=\overline{c}=\beta p_0$ (Eq. 21), and $H(z)=z^2+q_1$ is even. Then e factorizes as

$$e(z)=(z^2-\hat{s})(z^2-\hat{s}'),\qquad \hat{s}:=4\varepsilon_r a^2(p_0-p_1)-q_0,\qquad \hat{s}':=-4\varepsilon_r a^2(p_0+p_1)-q_0, \tag{41}$$

and we define

$$\hat{g}(z):=\frac{1}{z^2-\hat{s}},\qquad \hat{D}:=\beta-\frac{2\varepsilon_r a S_k(\hat{u})}{\hat{u}}, \tag{42}$$

where $\hat{u}$ is either square root of $\hat{s}$; $\hat{D}$ is even in $\hat{u}$. Explicitly, for even k,

$$\hat{D}=\beta\left[1+\frac{\varepsilon_r a}{\hat{u}}\tanh\frac{\beta\hat{u}}{2}\right], \tag{43}$$

with the value $\beta(1+\varepsilon_r a\beta/2)$ at $\hat{u}=0$. For odd k, coth replaces tanh, and $\hat{u}=0$ means $e(0)=0$ at the mirror fixed point $z_n=0$ (Proposition 6).

Corollary 5 (half filling) Lean. Let $\mu=0$, $e(z_n)\neq 0$ for all n, and $\hat{D}\neq 0$. Then

$$\Gamma_{r,nm}=\beta P(z_n)\left[\frac{1}{G_2(z_n)}-\frac{H(z_n)}{e(z_n)}\right]\delta_{nm}+2\varepsilon_r a-\lambda h F(z_n)\delta_{m\overline{n}}-\frac{2\varepsilon_r a\beta P(\hat{u})}{\hat{D}}\hat{g}(z_n)\hat{g}(z_m), \tag{44}$$

and the inverse χ_r^{-1} it represents is two-sided.

This is Eq. 19 of Ref. [1] for $r=\mathrm{d},\mathrm{m}$ (Appendix D). Compared with Eq. 40, the dense term has become a mirror term, and only one separable term with one thermal denominator remains. The corollary is

the limit $\mu \to 0$ of Theorem 4 (proof in Appendix B). Its assumption $e(z_n) \neq 0$ matters only where $\beta H(z_n) = -\lambda h$ with $n \neq \overline{n}$, a genuine divergence (Section 4.4); at all other zeros of $e(z_n)$ the formula has a removable singularity (Proposition 6).

4.3 Removable singularities

At isolated parameters a denominator of Eq. 34, Eq. 40 or Eq. 44 vanishes, but the entries of Γ_r stay finite. Let $O \coloneqq \{c, \overline{c}\}$ denote a mirror orbit; it has $|O| = 2$ elements, or $|O| = 1$ at a mirror fixed point.

Proposition 6 (removable singularities) Lean, in part. In the following cases the entries of Γ_r are the limits of the closed forms as the parameters approach the given values, and the inverse χ_r^{-1} they represent is two-sided.

1. *Singlet, $t_c + A_\mathrm{s} = 0$ for some c.* This happens on one orbit only, and K_s diverges. With

$$T' \coloneqq \sum_{l \notin O} \frac{1}{t_l + A_\mathrm{s}}, \qquad K' \coloneqq 1 + \frac{2aT'}{\beta}, \tag{45}$$

   the separable term of Eq. 34 drops out for $n, m \notin O$, and

$$\begin{aligned} \Gamma_{\mathrm{s},nm} &= 4a - \frac{2\beta \mathcal{P}(-A_\mathrm{s})}{|O|\ (t_m + A_\mathrm{s})} \quad (n \in O, m \notin O), \\ \Gamma_{\mathrm{s},nm} &= 4a + \frac{2\beta}{|O|}\left[\mathcal{P}'(-A_\mathrm{s}) - \frac{\mathcal{P}(-A_\mathrm{s})}{\mathcal{G}(-A_\mathrm{s})}\right] + \frac{\beta^2 \mathcal{P}(-A_\mathrm{s}) K'}{a\ |O|^2} \quad (n, m \in O), \end{aligned} \tag{46}$$

   and symmetrically for $n \notin O$, $m \in O$.
2. *Density and magnetic, $s_1 = s_2$* (the other assumptions of Theorem 4 holding). The sum over j in Eq. 40 is replaced by

$$\lambda\rho\, \partial_s \left[\frac{Z(s)}{(z_n^2 - s)(z_m^2 - s)}\right]_{s=s_1}, \qquad Z(s) \coloneqq \frac{Z(u) + Z(-u)}{2} \quad (u^2 = s), \tag{47}$$

   where $Z(u)$ is Z_j with u_j replaced by u.
3. *Density and magnetic, e vanishing on one orbit O* (at $\mu \neq 0$, the other assumptions of Theorem 4 holding). Eq. 40 holds, with the entries in the rows and columns of O taken as limits as μ approaches its value.
4. *Half filling, $\beta H(z_c) = \lambda h$,* i.e. $z_c^2 = \hat{s}$, with $\beta H(z_n) \neq -\lambda h$ for all $n \neq \overline{n}$. Then $\hat{u} = z_c$, and $\hat{D}$ diverges. With

$$\hat{T}' \coloneqq \sum_{l \notin O} \hat{g}(z_l), \qquad \hat{D}' \coloneqq \beta - 2\varepsilon_r a \hat{T}', \tag{48}$$

   the separable term of Eq. 44 drops out for $n, m \notin O$, and

$$\begin{aligned} \Gamma_{r,nm} &= 2\varepsilon_r a + \frac{\beta P(z_c)\hat{g}(z_m)}{|O|} \quad (n \in O, m \notin O), \\ \Gamma_{r,nm} &= \beta P(z_c)\left[\frac{\delta_{nm}}{G_2(z_c)} - \frac{\delta_{nm} - \delta_{m\overline{n}}}{2(\hat{s} - \hat{s}')}\right] + 2\varepsilon_r a + \frac{\beta P(z_c)\hat{D}'}{2\varepsilon_r a\ |O|^2} - \frac{2\beta(\hat{s} + q_0 - 2a^2)}{|O|} \quad (n, m \in O), \end{aligned} \tag{49}$$

   and symmetrically for $n \notin O$, $m \in O$. At a mirror fixed point with $\beta H(0) = -\lambda h$, Eq. 44 holds as a limit: its diagonal and mirror terms, which act on the same entry, stay finite together.

Item 1 is proved with Theorem 3, item 2 in Step 5 of the proof of Theorem 4, and items 3 and 4 in Appendix B.

4.4 Finite boxes and divergences

Proposition 7 (finite boxes). Under the assumptions of Theorem 2, Theorem 3, Theorem 4 or Corollary 5, or in the cases of Proposition 6, the restrictions of χ_r to the boxes $|2n+1+k| \le 2M+1$ are invertible for all sufficiently large M, and their inverses converge entrywise to $\chi_r^{-1} = \Gamma_r/\beta^2 + \eta_r \boldsymbol{B}^{-1}$.

Divergences. An entry of Γ_r can diverge only where a denominator in its closed form vanishes; Appendix G finds the null vectors and maps the divergences at $\omega = 0$. Apart from the removable cases of Proposition 6, and away from intersections of these exceptional sets, the remaining zeros are genuine poles under the following nonvanishing conditions:

- $K_j = 0$ with $u_j P(u_j) \neq 0$ and $s_1 \neq s_2$ (d, m). Then, as $\alpha_k(u_j) \neq 0$,
$$\gamma_j K_j = \frac{2\lambda\rho\beta u_j P(u_j) H(-u_j)}{s_j - s_\ell} \neq 0. \tag{50}$$
At $u_j = 0$ the zero of K_j can cancel, as at $\hat{u} = 0$ in Corollary 5.
- $\hat{D} = 0$ with $P(\hat{u}) \neq 0$ (d, m at $\mu = 0$).
- $\beta H(z_n) = -\lambda h$ with $n \neq \overline{n}$ (d, m at $\mu = 0$). There $N(z,\cdot)/P$ is even, so by Eq. 104
$$P(z)\chi_r(\boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}) = -(\beta H(z_n) + \lambda h)(\boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}) = 0. \tag{51}$$
So the mirror-odd vector is a null vector. For $r = \mathrm{d}$, $k = 0$, $n = 0$ this is $\beta U = 2\pi/\sqrt{3}$.
- $K_{\mathrm{s}} = 0$ with $\mathcal{P}(-A_{\mathrm{s}}) \neq 0$ (s).
- $t_n + a\rho b + a^2 = 0$ with $n \neq \overline{n}$ (t); the numerator is $a^2(1-\rho^2)\mathcal{P}(t_n) \neq 0$.

Example. *Closed form against finite boxes* ($r = \mathrm{d}$, $U = 2$, $\beta = 3$, $\mu = 0.3$, $k = 1$). The inverses of χ_{d} on the boxes of Proposition 7 ($2M+1$ frequencies, as k is odd) converge like $1/M$; the extrapolation $2\Gamma^{(256)} - \Gamma^{(128)}$ removes the leading error.

(n,m)	$M = 128$	$M = 256$	extrapolated	Eq. 40
$(0,0)$	$25.9568 + 33.8867i$	$25.9592 + 33.8867i$	$25.96155 + 33.88673i$	$25.96157 + 33.88673i$
$(0,\overline{0})$	-16.2893	-16.2870	-16.28460	-16.28459
$(2,-1)$	$2.2186 + 0.0612i$	$2.2210 + 0.0612i$	$2.22334 + 0.06123i$	$2.22336 + 0.06123i$

5 Matsubara sums

Lemma 8 (basic sum) Lean, in part. For x off the lattice, with S_k from Eq. 28,
$$\mathrm{PV}\sum_n \frac{1}{z_n - x} = S_k(x), \qquad \sum_n \frac{1}{(z_n - x)^j} = \frac{S_k^{(j-1)}(x)}{(j-1)!} \quad (j \ge 2), \tag{52}$$
where the second sum converges absolutely. Moreover, for all k,
$$S_k(x \pm w/2) = -\frac{\beta}{2}\tanh(\beta(x-\mu)/2). \tag{53}$$
In particular, S_k takes the value $S_\pm := \pm(\beta/2)\tanh(\beta(a \pm \mu)/2)$ at both zeros $\mp a \pm w/2$ of $Q_\pm$; by Eq. 4,
$$S_+ = \frac{\beta(p_1 - p_0)}{2(p_1 + p_0)}, \qquad S_- = \frac{\beta(p_2 - p_1)}{2(p_1 + p_2)}. \tag{54}$$

Proof. Since $2n+1+k$ is odd for even k and even for odd k,

$$z_n - x = \begin{cases} i\nu_{n'} - (x-\mu) & k \text{ even}, \quad n' := n + k/2 \\ i\Omega_{n''} - (x-\mu) & k \text{ odd}, \quad n'' := (2n+1+k)/2 \end{cases} \tag{55}$$

and the box is symmetric in n' (resp. n''). The symmetric Mittag-Leffler expansions

$$\tanh y = \sum_j \frac{1}{y - i\pi\left(j + \frac{1}{2}\right)}, \qquad \coth y = \sum_j \frac{1}{y - i\pi j} \tag{56}$$

at $y = \beta(x-\mu)/2$ read

$$\tanh \frac{\beta(x-\mu)}{2} = \frac{2}{\beta} \sum_j \frac{1}{x - \mu - i\nu_j}, \qquad \coth \frac{\beta(x-\mu)}{2} = \frac{2}{\beta} \sum_j \frac{1}{x - \mu - i\Omega_j}, \tag{57}$$

which is Eq. 28. For $j \geq 2$ the $(j-1)$-fold x-derivative of the series converges absolutely and uniformly on compact sets away from the lattice, so it may be differentiated term by term. For Eq. 53: $w/2 = i\pi k/\beta$ shifts the argument of tanh or coth by $\pm i\pi k/2$. For even k this is a multiple of the period $i\pi$; for odd k use $\coth(y \pm i\pi/2) = \tanh y$. □

Lemma 9 (residue rule). Let φ be proper rational without poles on the lattice. Then

$$\sum_n \varphi(z_n) = \sum_{x \in \text{ poles}(\varphi)} \operatorname*{Res}_{t=x} [\varphi(t) S_k(t)], \tag{58}$$

absolutely convergent if $\varphi = O(t^{-2})$, and as a PV sum if $\varphi \sim t^{-1}$. ($\operatorname{Res}_{t=x}$ of a function is the coefficient of $(t-x)^{-1}$ in its Laurent expansion at x.)

Proof. Partial fractions write φ as a finite sum of constants times $(t-x)^{-j}$, $j \geq 1$, over its poles x. By Lemma 8 each $(t-x)^{-j}$ sums to $S_k^{(j-1)}(x)/(j-1)!$, which is the residue of $(t-x)^{-j} S_k(t)$ at x, since S_k is analytic there. □

Lemma 10 (particle–particle sum) Lean. For $t \neq t_n$ for all n, the sum converges absolutely to $\mathcal{T}_k$ of Eq. 30,

$$\sum_n \frac{1}{t_n - t} = \mathcal{T}_k(t). \tag{59}$$

At the two points $t_\pm := -a^2 \pm ab$, for $b \neq \pm 2a$,

$$\mathcal{T}_k(t_+) = -\frac{2S_-}{2a-b}, \qquad \mathcal{T}_k(t_-) = \frac{2S_+}{2a+b}. \tag{60}$$

If $b = \pm 2a$ (which needs $k = 0$), then $t_\pm = b^2/4$, i.e. $\xi = 0$, and $\mathcal{T}_k(t_\pm) = \beta^2/4$.

Proof. Since $x_n = z_n - w/2$ and $b/2 = \mu - w/2$, $t_n = b^2/4 - (z_n - \mu)^2$. So for $\xi \neq 0$, by Lemma 8 and the oddness of tanh and coth,

$$\sum_n \frac{1}{t_n - t} = \frac{1}{2\xi} \sum_n \left[\frac{1}{z_n - \mu + \xi} - \frac{1}{z_n - \mu - \xi} \right] = \frac{S_k(\mu - \xi) - S_k(\mu + \xi)}{2\xi} = -\frac{S_k(\mu + \xi)}{\xi}; \tag{61}$$

the sum converges absolutely, as $t_n \sim n^2$. For $\xi = 0$, $(t_n - t)^{-1} = -(z_n - \mu)^{-2}$ sums to $-S_k'(\mu) = \beta^2/4$ by Lemma 8. This occurs only for even k: for odd k, $t = b^2/4$ is t_n at the mirror fixed point.

At $t_\pm$, $\xi^2 = (a \mp b/2)^2$. At t_+ take $\xi = a - b/2$: then $\mu + \xi = a + w/2$, and $S_k(\mu + \xi) = S_-$ by Eq. 53. At t_- take $\xi = -a - b/2$: then $\mu + \xi = -a + w/2$, and $S_k(\mu + \xi) = S_+$. □

6 Proofs for the singlet and triplet channels

6.1 Variables

With $t_\pm$ from Lemma 10, $\mathcal{P}(t) = (t - t_+)(t - t_-)$. Define

$$d_\pm := A_\mathrm{s} + t_\pm, \qquad c := c_{-k} = \begin{cases} \rho/b & (\mu, k) \neq (0,0) \\ \beta p_0 & \mu = k = 0 \end{cases}. \tag{62}$$

Then $d_+ d_- = \mathcal{P}(-A_\mathrm{s})$. For $t = x(b-x)$, direct expansion gives

$$\begin{gathered} (x-a)(b-x-a) = t - t_+, \qquad (x+a)(b-x+a) = t - t_-, \\ (x + a\rho)(b - x + a\rho) = \mathcal{G}(t), \end{gathered} \tag{63}$$

and the product of the first two is $(x^2 - a^2)((b-x)^2 - a^2) = \mathcal{P}(t)$. Hence, by Eq. 15,

$$B_n = -\frac{\beta}{2}\frac{\mathcal{G}(t_n)}{\mathcal{P}(t_n)} = B_{\overline{n}}. \tag{64}$$

From particle–hole to particle–particle. The pp entry (n, m) at transfer k is the ph entry at the bosonic index and frequency

$$k' := -k - n - m - 1, \qquad w' := i\omega_{k'} = b - x_n - x_m. \tag{65}$$

Then $\delta_{k'0} = \delta_{m\overline{n}}$, and the pair coefficient becomes $c_{n+m+k'+1} = c$. In the variables of Lemma 1 at transfer w', $z + v = x_n + x_m + w' = b$. By Eq. 23 and Eq. 63,

$$Q_-(z)Q_-(v) = (x_n - a)(b - x_n - a)(x_m - a)(b - x_m - a) = (t_n - t_+)(t_m - t_+), \tag{66}$$

and likewise $Q_+(z)Q_+(v) = (t_n - t_-)(t_m - t_-)$.

6.2 Structure

A diagonal matrix whose entries depend on n only through t_n commutes with the mirror $\mathcal{M}$. Define

$$\begin{aligned} &\mathcal{M}_\pm := (\mathbb{1} \pm \mathcal{M})/2, && \text{projectors onto mirror-even and mirror-odd vectors} \\ &L_\mathrm{t}(t) := -\frac{\beta}{2}\frac{t + a\rho b + a^2}{\mathcal{P}(t)}, \quad L_\mathrm{s}(t) := \frac{\beta}{2}\frac{t + A_\mathrm{s}}{\mathcal{P}(t)}, \\ &g_\pm(t) := \frac{1}{t - t_\pm}, \quad \gamma_\pm := \frac{a(\rho \pm 1)(2a \mp b)}{2b}, \end{aligned} \tag{67}$$

and $\boldsymbol{L}_{\mathrm{s,\,t}} := \mathrm{diag}\big(L_{\mathrm{s,\,t}}(t_n)\big)$.

Proposition 11 (pp) Lean.

$$\chi_\mathrm{t} = \boldsymbol{B}\mathcal{M}_+ + \boldsymbol{L}_\mathrm{t}\mathcal{M}_-, \qquad \chi_\mathrm{s} = \boldsymbol{L}_\mathrm{s}\mathcal{M}_+ - \boldsymbol{B}\mathcal{M}_- + \chi_{\mathrm{s,\,reg}}, \tag{68}$$

where

$$\big(\chi_{\mathrm{s,\,reg}}\big)_{nm} = \frac{a(2ac-1)(t_n - t_-)(t_m - t_-) + a^2(1-\rho)(b+2a)(t_n + t_m + 2a^2)}{\mathcal{P}(t_n)\mathcal{P}(t_m)}, \tag{69}$$

which for $b \neq 0$ equals

$$\big(\chi_{\mathrm{s,\,reg}}\big)_{nm} = \gamma_+ g_+(t_n)g_+(t_m) + \gamma_- g_-(t_n)g_-(t_m). \tag{70}$$

Proof. Insert $w \to w'$ into Eq. 18 and Eq. 19, using

$$x_n + w' = b - x_m = x_{\overline{n}} \quad (\delta_{nm}\text{ terms}), \qquad x_m = x_{\overline{n}}, \quad x_n + w' = x_n \quad (\delta_{k'0} = \delta_{m\overline{n}}\text{ terms}). \tag{71}$$

With Eq. 63 (abbreviating $\mathcal{P} \coloneqq \mathcal{P}(t_n)$),

$$\begin{aligned}\chi_{\mathrm{pp},\uparrow\uparrow} &= 2B_n\delta_{nm} - \frac{\beta a^2(1-\rho^2)}{\mathcal{P}}(\delta_{nm} - \delta_{m\overline{n}}),\\ \chi_{\mathrm{pp},\uparrow\downarrow} &= -\frac{4\beta a^2 p_1}{\mathcal{P}}\delta_{nm} + \frac{4\beta a^2(p_0p_2 - p_1^2)}{\mathcal{P}}\delta_{m\overline{n}} + \mathcal{N}|_{w=w'}.\end{aligned} \tag{72}$$

Triplet. By Eq. 64,

$$\frac{\chi_{\mathrm{pp},\uparrow\uparrow} + 2\boldsymbol{B}}{4} = \left[B_n - \frac{\beta a^2(1-\rho^2)}{4\mathcal{P}}\right]\delta_{nm} + \frac{\beta a^2(1-\rho^2)}{4\mathcal{P}}\delta_{m\overline{n}} = \frac{B_n + L_{\mathrm{t}}}{2}\delta_{nm} + \frac{B_n - L_{\mathrm{t}}}{2}\delta_{m\overline{n}}, \tag{73}$$

with $L_{\mathrm{t}} = L_{\mathrm{t}}(t_n)$. As $\boldsymbol{B}$ and $\boldsymbol{L}_{\mathrm{t}}$ commute with $\mathcal{M}$, this is $\boldsymbol{B}\mathcal{M}_+ + \boldsymbol{L}_{\mathrm{t}}\mathcal{M}_-$.

Singlet, delta terms. The δ_{nm} and $\delta_{m\overline{n}}$ terms of $\big(-\chi_{\mathrm{pp},\uparrow\uparrow} + 2\chi_{\mathrm{pp},\uparrow\downarrow} - 2\boldsymbol{B}\big)/4$ are

$$\left[-B_n + \frac{\beta a^2(1-\rho^2-8p_1)}{4\mathcal{P}}\right]\delta_{nm} + \frac{\beta a^2[8(p_0p_2 - p_1^2) - (1-\rho^2)]}{4\mathcal{P}}\delta_{m\overline{n}}. \tag{74}$$

The first coefficient is $(L_{\mathrm{s}} - B_n)/2$. By Eq. 6 and $p_0 + p_2 = 1 - 2p_1$, the bracket of the second is $1 - 8p_1 - \rho^2$, so it is $(L_{\mathrm{s}} + B_n)/2$. Together: $\boldsymbol{L}_{\mathrm{s}}\mathcal{M}_+ - \boldsymbol{B}\mathcal{M}_-$.

Singlet, remaining terms. They are $\mathcal{N}|_{w=w'}\ /2$. In Eq. 24 at transfer w',

$$4a^2C_{nm} \to 4a^2c, \qquad z + v = b, \qquad zv + q_0 = a^2 + (t_n + t_m)/2 \quad \text{(identity (A3))}, \tag{75}$$

and the denominators are Eq. 66. This gives Eq. 69; for $b \neq 0$, its partial fractions in t_n and t_m are Eq. 70 (identity (A3)). □

6.3 Proofs

Proof of Theorem 2. χ_{t} and $\boldsymbol{B}$ act on each orbit $\{n, \overline{n}\}$ separately. For $n \neq \overline{n}$ the orbit carries a 2×2 block with eigenvalue B_n on the even vector and $L_{\mathrm{t}}(t_n)$ on the odd vector; $L_{\mathrm{t}}(t_n) \neq 0$ is the assumption $t_n + a\rho b + a^2 \neq 0$, which must hold on every orbit for χ_{t} to be invertible. So

$$\chi_{\mathrm{t}}^{-1} = \boldsymbol{B}^{-1}\mathcal{M}_+ + \boldsymbol{L}_{\mathrm{t}}^{-1}\mathcal{M}_-, \qquad \Gamma_{\mathrm{t}} = \beta^2(\boldsymbol{L}_{\mathrm{t}}^{-1} - \boldsymbol{B}^{-1})\mathcal{M}_- \tag{76}$$

by Eq. 16. With $\mathcal{G} - (t + a\rho b + a^2) = -a^2(1-\rho^2)$,

$$\frac{\beta^2}{2}\left(\frac{1}{L_{\mathrm{t}}} - \frac{1}{B}\right) = -\frac{\beta\mathcal{P}[\mathcal{G} - (t + a\rho b + a^2)]}{(t + a\rho b + a^2)\mathcal{G}} = \frac{\beta a^2(1-\rho^2)\mathcal{P}}{(t + a\rho b + a^2)\mathcal{G}}, \tag{77}$$

which is Eq. 32. At a fixed point the orbit is $\{n\}$ with $\mathcal{M}_- = 0$; then $\chi_{\mathrm{t},nn} = B_n$, $\Gamma_{\mathrm{t},nn} = 0$, and $L_{\mathrm{t}}(t_n)$ does not enter. □

Proof of Theorem 3. Idea. χ_{s} is an easily invertible matrix plus the rank-two term Eq. 70, so the Woodbury formula applies. Its 2×2 matrix becomes diagonal in a suitable basis.

For $U = 0$ both sides vanish ($L_{\mathrm{s}} = -B$, $\chi_{\mathrm{s,\,reg}} = 0$). Let $U \neq 0$, and first assume $b \neq 0$, $b \neq \pm 2a$ and $d_\pm \neq 0$. Then $\gamma_\pm \neq 0$, since $|\rho| < 1$.

Woodbury. Let $\boldsymbol{g}$ be the $\infty \times 2$ matrix with rows $(g_+(t_n), g_-(t_n))$ and $\gamma \coloneqq \mathrm{diag}(\gamma_+, \gamma_-)$, so that

$$\chi_{\mathrm{s,\,reg}} = \boldsymbol{g}\gamma\boldsymbol{g}^\top, \qquad \mathcal{M}_+\boldsymbol{g} = \boldsymbol{g}. \tag{78}$$

The matrix

$$\boldsymbol{X} \coloneqq \boldsymbol{L}_{\mathrm{s}}\mathcal{M}_+ - \boldsymbol{B}\mathcal{M}_- \quad \text{has the inverse} \quad \boldsymbol{X}^{-1} = \boldsymbol{L}_{\mathrm{s}}^{-1}\mathcal{M}_+ - \boldsymbol{B}^{-1}\mathcal{M}_- \tag{79}$$

(all products are finite sums). Set

$$\boldsymbol{V}_{\mathrm{s}} \coloneqq \boldsymbol{X}^{-1}\boldsymbol{g} = \boldsymbol{L}_{\mathrm{s}}^{-1}\boldsymbol{g}, \qquad \Psi \coloneqq \boldsymbol{g}^{\top}\boldsymbol{V}_{\mathrm{s}} = \sum_{n} \frac{(g_+, g_-)^{\top}(g_+, g_-)}{L_{\mathrm{s}}}(t_n), \tag{80}$$

absolutely convergent (terms $O(t_n^{-1}) = O(n^{-2})$). If $\gamma^{-1} + \Psi$ is invertible, then

$$Y \coloneqq \boldsymbol{X}^{-1} - \boldsymbol{V}_{\mathrm{s}}(\gamma^{-1} + \Psi)^{-1}\boldsymbol{V}_{\mathrm{s}}^{\top} \tag{81}$$

satisfies

$$\chi_{\mathrm{s}} Y = \mathbb{1} + \boldsymbol{g}\Big[\gamma - (\mathbb{1} + \gamma\Psi)(\gamma^{-1} + \Psi)^{-1}\Big]\boldsymbol{V}_{\mathrm{s}}^{\top} = \mathbb{1}. \tag{82}$$

Y is symmetric, so also $Y\chi_{\mathrm{s}} = \mathbb{1}$. By Eq. 16 ($\eta_{\mathrm{s}} = -1$),

$$\Gamma_{\mathrm{s}} = \beta^2(\boldsymbol{L}_{\mathrm{s}}^{-1} + \boldsymbol{B}^{-1})\mathcal{M}_{+} - \beta^2\boldsymbol{V}_{\mathrm{s}}(\gamma^{-1} + \Psi)^{-1}\boldsymbol{V}_{\mathrm{s}}^{\top}. \tag{83}$$

The first term gives the first term of Eq. 34, since $\mathcal{G} - t - A_{\mathrm{s}} = a^2(\rho^2 - 1 + 8p_1)$ and

$$\frac{\beta^2}{2}\left(\frac{1}{L_{\mathrm{s}}} + \frac{1}{B}\right) = \beta\mathcal{P}\left[\frac{1}{t + A_{\mathrm{s}}} - \frac{1}{\mathcal{G}}\right]. \tag{84}$$

Change of basis. Let $\kappa_n \coloneqq 1/(t_n + A_{\mathrm{s}})$. Since $\mathcal{P} = (t - t_+)(t - t_-)$ and $t - t_{\mp} = (t + A_{\mathrm{s}}) - d_{\mp}$,

$$\frac{g_{\pm}(t_n)}{L_{\mathrm{s}}(t_n)} = \frac{2}{\beta}(1 - d_{\mp}\kappa_n), \quad \text{i.e.} \quad (\boldsymbol{V}_{\mathrm{s}})_{n,:}^{\top} = \frac{2}{\beta}\,\Theta^{\top}(1, \kappa_n)^{\top}, \quad \Theta \coloneqq \begin{pmatrix} 1 & 1 \\ -d_- & -d_+ \end{pmatrix}. \tag{85}$$

Hence

$$\Big(\boldsymbol{V}_{\mathrm{s}}(\gamma^{-1} + \Psi)^{-1}\boldsymbol{V}_{\mathrm{s}}^{\top}\Big)_{nm} = \left(\frac{2}{\beta}\right)^2 (1, \kappa_n)\Xi^{-1}(1, \kappa_m)^{\top}, \qquad \Xi \coloneqq \Theta^{-\top}(\gamma^{-1} + \Psi)\Theta^{-1}. \tag{86}$$

Insert

$$\Psi = \sum_{n} \left(\frac{2}{\beta}\right)^2 L_{\mathrm{s}}(t_n)\Theta^{\top}(1, \kappa_n)^{\top}(1, \kappa_n)\Theta, \qquad \left(\frac{2}{\beta}\right)^2 L_{\mathrm{s}} = \frac{2}{\beta}\frac{t + A_{\mathrm{s}}}{\mathcal{P}}. \tag{87}$$

Then

$$\Xi = \Theta^{-\top}\gamma^{-1}\Theta^{-1} + \frac{2}{\beta}\sum_{n} \frac{1}{\mathcal{P}(t_n)} \begin{pmatrix} t_n + A_{\mathrm{s}} & 1 \\ 1 & 1/(t_n + A_{\mathrm{s}}) \end{pmatrix}. \tag{88}$$

By partial fractions over $t_+, t_-, -A_{\mathrm{s}}$, the three sums are combinations of $\mathcal{T}_k(t_+)$, $\mathcal{T}_k(t_-)$ and $\mathcal{T}_k(-A_{\mathrm{s}})$. Inserting Eq. 60, the off-diagonal entries of the two terms cancel, and (identity (A4))

$$\Xi = \mathrm{diag}\left(-\frac{1}{a}, \frac{K_{\mathrm{s}}}{a d_+ d_-}\right). \tag{89}$$

So $\gamma^{-1} + \Psi$ is invertible if and only if $K_{\mathrm{s}} \neq 0$, and

$$-\beta^2\left(\frac{2}{\beta}\right)^2 (1, \kappa_n)\Xi^{-1}(1, \kappa_m)^{\top} = -4\left[-a + \frac{a d_+ d_- \kappa_n \kappa_m}{K_{\mathrm{s}}}\right] = 4a - \frac{4a\,\mathcal{P}(-A_{\mathrm{s}})}{K_{\mathrm{s}}(t_n + A_{\mathrm{s}})(t_m + A_{\mathrm{s}})}. \tag{90}$$

Excluded values. Fix k. The values of μ where $b = 0$, $b = \pm 2a$ or $d_{\pm} = 0$, and those where $t_n + A_{\mathrm{s}} = 0$ for some n (Proposition 6), are isolated, since these are zeros of nonconstant analytic functions of μ. Eq. 34 contains no b, $2a \pm b$ or $d_{\pm}$ in a denominator, so it is continuous at the first kind.

Lattice zeros ($t_n + A_{\mathrm{s}} \to 0$; item 1 of Proposition 6). Put $q \coloneqq t_n + A_{\mathrm{s}}$, and let $O \coloneqq \{n, \overline{n}\}$, on which t_n is constant. Then

$$\mathcal{T}_k(-A_{\mathrm{s}}) = \frac{|O|}{q} + T' + O(q), \qquad K_{\mathrm{s}} = \frac{2a\ |O|}{\beta q} + O(1). \tag{91}$$

Since $a^2(\rho^2 - 1 + 8p_1) = \mathcal{G}(t_n) - q$ and $\mathcal{G}(t_n) \neq 0$ (Eq. 63), the first and the last term of Eq. 34 are

$$\frac{\beta\mathcal{P}(-A_{\mathrm{s}})}{q}(\delta_{nm} + \delta_{m\overline{n}}) + O(1), \qquad -\frac{2\beta\mathcal{P}(-A_{\mathrm{s}})}{|O|\ q} + O(1) \quad (m \in O), \tag{92}$$

and the last one is $O(1)$ for $m \notin O$. For $m \in O$, $\delta_{nm} + \delta_{m\overline{n}} = 2/\ |O|$, so the $1/q$ terms cancel. The next order, with $\mathcal{P}(t_n) = \mathcal{P}(-A_{\mathrm{s}}) + q\mathcal{P}'(-A_{\mathrm{s}}) + O(q^2)$, gives Eq. 46.

Two-sidedness at the excluded values. Near all these values, $(\chi_{\mathrm{s}})_{nl} = O(|t_l|^{-1})$ and Y_{lm} is bounded for $l \notin \{n, \overline{n}, m, \overline{m}\}$, uniformly, where $Y = \Gamma_{\mathrm{s}}/\beta^2 - \boldsymbol{B}^{-1}$ with Eq. 34. As $|t_l| \sim l^2$, Lemma 18 extends $\chi_{\mathrm{s}} Y = \mathbb{1}$ to these values. □

7 The Cauchy matrix

Throughout this section $\mu \neq 0$.

Lemma 12 (values on the reflected lattice) Lean, in part. For every n,

$$S_k(-z_n) = \frac{\beta}{2}\coth(\beta\mu), \qquad S_k'(-z_n) = \frac{\beta^2}{4\sinh^2(\beta\mu)}, \qquad \rho\, S_k(-z_n) = \overline{c}. \tag{93}$$

Proof. Write $-z_n = x - w/2$ with $x := -\mu - i\pi(2n+1)/\beta$. By Eq. 53,

$$S_k(-z_n) = -\frac{\beta}{2}\tanh(-\beta\mu - i\pi(2n+1)/2) = \frac{\beta}{2}\coth(\beta\mu), \tag{94}$$

as tanh has period $i\pi$ and $\tanh(y - i\pi/2) = \coth y$. Differentiating Eq. 53 in x gives

$$S_k'(-z_n) = -\frac{\beta^2}{4}\cosh^{-2}(-\beta\mu - i\pi(2n+1)/2) = \frac{\beta^2}{4\sinh^2(\beta\mu)}. \tag{95}$$

The last identity is Eq. 7. □

So S_k is *constant* on the reflected lattice (Figure 2). This is the source of all simplifications below.

Lemma 13 (square) Lean. $\sum_l C_{nl}C_{lm} = h^2\delta_{nm}$, absolutely convergent.

Proof. $C_{nl}C_{lm} = O(|z_l|^{-2})$. For $n \neq m$, partial fractions in z_l and Lemma 8 give

$$\sum_l C_{nl}C_{lm} = \frac{\rho^2}{z_m - z_n}\sum_l\left[\frac{1}{z_l + z_n} - \frac{1}{z_l + z_m}\right] = \rho^2\frac{S_k(-z_n) - S_k(-z_m)}{z_m - z_n} = 0 \tag{96}$$

by Lemma 12. For $n = m$, by Lemma 12 and Eq. 7,

$$\sum_l C_{nl}C_{ln} = \rho^2\sum_l \frac{1}{(z_l + z_n)^2} = \rho^2 S_k'(-z_n) = h^2. \tag{97}$$

□

The action of C on rational functions is governed by the thermal function α_k of Eq. 38.

Lemma 14 (thermal function) Lean. With $y_\pm := \beta(x \pm \mu)/2$,

$$\alpha_k(x) = \begin{cases} -h\cosh(y_+)/\cosh(y_-) & k \text{ even} \\ -h\sinh(y_+)/\sinh(y_-) & k \text{ odd} \end{cases}, \qquad \alpha_k(x)\alpha_k(-x) = h^2. \tag{98}$$

Proof. For even k, insert $S_k(x) = -(\beta/2)\tanh y_-$, $p_2 = p_0 e^{2\beta\mu}$ and $h = \beta p_0 e^{\beta\mu}$:

$$\alpha_k(x) = -\frac{\beta}{2}\frac{p_2 e^{y_-} + p_0 e^{-y_-}}{\cosh y_-} = -h\frac{\cosh(y_- + \beta\mu)}{\cosh y_-}, \quad y_- + \beta\mu = y_+. \tag{99}$$

Under $x \to -x$, $y_\pm \to -y_\mp$, so the two quotients multiply to 1. Odd k: replace tanh, cosh by coth, sinh. □

Let $\mathcal{C}$ be the action of C on functions sampled on the lattice, $(\mathcal{C}\,\varphi)(z_m) := \sum_n C_{mn}\varphi(z_n)$.

Lemma 15 (Cauchy calculus) Lean, in part. For x with x and $-x$ off the lattice, and every lattice point z,

$$\mathcal{C}[(t-x)^{-1}](z) = \frac{\alpha_k(x)}{z+x}, \qquad \mathcal{C}[1](z) = \overline{c} \quad \text{(PV)}. \tag{100}$$

Higher poles follow by applying $(1/j!)\partial_x^j$; so $\mathcal{C}$ maps every proper rational function whose poles x satisfy this condition to an explicit proper rational function.

Proof. Partial fractions and Lemma 8 give

$$\sum_n \frac{\rho}{(z+z_n)(z_n-x)} = \frac{\rho[S_k(x) - S_k(-z)]}{z+x} = \frac{\alpha_k(x)}{z+x} \tag{101}$$

by Lemma 12. The constant rule is $\mathrm{PV}\sum_n C_{mn} = \rho S_k(-z_m) = \overline{c}$. The sums of x-derivatives converge uniformly near any admissible x, so Eq. 100 may be differentiated. □

8 Proofs for the density and magnetic channels

8.1 Structure

Let $\delta := \delta_{k0}$ and

$$\zeta_r := \beta a^2[1 - \rho^2 + 4\varepsilon_r(p_0p_2 - p_1^2)], \qquad N_2(v) := -2\varepsilon_r a(v^2 + 2a\rho v + q_0) + \delta\zeta_r(v^2 - a^2), \tag{102}$$

$$N_0(v) := (v^2 + 2q_0 - 4a^2)N_2(v) + (2\varepsilon_r a - \delta\zeta_r)P(v), \qquad N(z,v) := N_0(v) + z^2 N_2(v). \tag{103}$$

P is even by Eq. 23 and has no zeros on the lattice. In N_0 the terms of degree four cancel, so $N_0(v)/P(v) = O(v^{-1})$ and $N_2(v)/P(v) = O(v^{-2})$. We identify a function φ on the lattice with the vector $(\varphi(z_n))_n$ and write $(\chi_r\varphi)(z_n) := \sum_m \chi_{r,nm}\varphi(z_m)$.

Proposition 16 (structure) Lean.

$$P(z_n)\chi_{r,nm} = -\beta H(z_n)\delta_{nm} + \lambda C_{nm} + \frac{N(z_n, z_m)}{P(z_m)}. \tag{104}$$

Proof. Write $z := z_n$, $v := z_m$, and collect the terms of $\chi_r = \chi_{\uparrow\uparrow} + \varepsilon_r\chi_{\uparrow\downarrow}$ (Eq. 18, Eq. 19).

(a) The δ_{nm} terms are

$$-\beta[G(x_n)G(x_n+w) + a^2(1 - \rho^2 + 4\varepsilon_r p_1)J(x_n)J(x_n+w)]. \tag{105}$$

By Eq. 11 and Eq. 23,

$$G(x_n)G(x_n+w)=\frac{(z+a\rho)^2-w^2/4}{P(z)},\qquad J(x_n)J(x_n+w)=\frac{1}{P(z)},\tag{106}$$

and the numerators add up to $(z+a\rho)^2-w^2/4+a^2(1-\rho^2)+4\varepsilon_r a^2 p_1=H(z)$.

(b) The δ_{k0} terms occur only for $k=0$, where $w=0$, $x_n=z$ and $P(z)=\left(z^2-a^2\right)^2$. They are $\zeta_r J(z)J(v)$, and $P(z)J(z)J(v)=(z^2-a^2)(v^2-a^2)/P(v)$.

(c) The remaining terms are $\varepsilon_r\mathcal{N}_{nm}$ with Eq. 24. Since $C_{nm}(z+v)=\rho$ and $Q_+(z)-Q_-(v)=(z+v)(z-v+2a)$,

$$Q_+(z)C_{nm}=Q_-(v)C_{nm}+\rho(z-v+2a).\tag{107}$$

With $1/Q_-(v)=Q_+(v)/P(v)$ this gives

$$\begin{aligned}P(z)\mathcal{N}_{nm}=4a^2C_{nm}+\frac{1}{P(v)}[&4a^2\rho(z-v+2a)Q_+(v)-2aQ_+(z)Q_+(v)\\&+4a^2(1-\rho)(z+v+2a)(zv+q_0)].\end{aligned}\tag{108}$$

The bracket is quadratic in z. Using $v^2+2av+q_0=Q_+(v)$, its coefficients are

$$\begin{aligned}z^1:&\quad 4a^2Q_+(v)[\rho-1+(1-\rho)]=0,\\ z^2:&\quad -2aQ_+(v)+4a^2(1-\rho)v=-2a(v^2+2a\rho v+q_0),\\ z^0:&\quad -2a(v^2+2q_0-4a^2)(v^2+2a\rho v+q_0)+2aP(v)\quad\text{(identity (A2))}.\end{aligned}\tag{109}$$

Times ε_r, this is the part of Eq. 103 without ζ_r. The term (b) is the rest, since $(v^2-2a^2)(v^2-a^2)-P(v)=-a^2(v^2-a^2)$ at $k=0$. □

So, after multiplication by P, the finite-rank part of χ_r has only two output directions, 1 and z^2.

8.2 Rank-two sums

Lemma 17 (rank-two sums) Lean. For x off the lattice with $P(x)\neq 0$, and every z,

$$\mathcal{I}(z;x):=\sum_l\frac{N(z,z_l)}{P(z_l)(z_l-x)}=\overline{\mathcal{I}}(z;x)+\alpha_k(x)\frac{N(z,x)}{\rho P(x)},\tag{110}$$

absolutely convergent, where

$$\overline{\mathcal{I}}(z;x):=\frac{\overline{c}N(z,x)}{\rho P(x)}+\sum_{y:P(y)=0}\operatorname*{Res}_{t=y}\frac{S_k(t)N(z,t)}{P(t)(t-x)}.\tag{111}$$

$\overline{\mathcal{I}}$ depends on the temperature only through β, $\overline{c}$ and $S_\pm$ (Lemma 8), and $P(x)\overline{\mathcal{I}}(z;x)$ is a polynomial in z and x.

Proof. At the zeros of P, $S_k=S_\pm$ (Lemma 8); for $k=0$ they are double, and there $S_0'=S_0^2-\beta^2/4$ by Eq. 53. The summand of $\mathcal{I}$ is $O(z_l^{-2})$, since N_0 has degree at most three. Lemma 9 gives the residue $S_k(x)N(z,x)/P(x)$ at $t=x$, plus the residues at the zeros of P; insert $\rho S_k(x)=\overline{c}+\alpha_k(x)$ (Eq. 38). In x, $\overline{\mathcal{I}}$ has poles only at the zeros of P, of at most their multiplicity. □

8.3 The proof of Theorem 4

Strategy. For $U=0$, $\chi_r=\boldsymbol{B}$ and $\Gamma_{\mathrm{d}}=\Gamma_{\mathrm{m}}=0$; let $U\neq 0$. Fix a lattice point $v=z_m$. We construct a bounded function y_v on the lattice with $\chi_r y_v=\boldsymbol{e}_m$, the m-th unit vector. By Eq. 104 this means, for every lattice point z,

$$-\beta H(z)y_v(z) + \lambda(\mathcal{C}\, y_v)(z) + \sum_l \frac{N(z,z_l)}{P(z_l)} y_v(z_l) = P(v)\delta_{zv}. \tag{112}$$

For $l \neq n$, $\chi_{r,nl} = O(|z_l|^{-2})$ by Eq. 18, Eq. 19 and Eq. 24, so $\sum_l \chi_{r,nl} y_v(z_l)$ converges absolutely. Its parts in Eq. 112 converge at least as PV sums (Lemma 15, Lemma 9), so they may be evaluated separately.

We use Lemma 17 at $x = -v$, and Lemma 15 and Lemma 17 at $x = \pm u_j$. These points are off the lattice ($\mu \neq 0$, $e(z_n) \neq 0$), and $P(-v) = P(v) \neq 0$. We first assume that $P(u_j) \neq 0$ for the non-atomic pairs; Step 5 removes this assumption.

Step 1 (the Cauchy column). Through C, a unit vector at v produces the column $\lambda\rho/(z+v)$. Conversely, C maps $1/(t+v)$ back to a unit vector: by Lemma 13,

$$\mathcal{C}[(t+v)^{-1}](z) = \rho^{-1} \sum_l C(z,z_l)C(z_l,v) = (h^2/\rho)\delta_{zv}. \tag{113}$$

So try $y = D_v\delta_{zv} + E_v/(z+v)$. Insert it into the left side of Eq. 112, using $H(z) = H(-v) + (z+v)(z-v+2a\rho)$, and Eq. 110 at $x = -v$, where $\alpha_k(-v) = 0$ (Lemma 12). The left side becomes

$$\begin{aligned} &\left[-\beta H(v)D_v + \frac{\lambda h^2}{\rho}E_v\right]\delta_{zv} + \frac{\lambda\rho D_v - \beta H(-v)E_v}{z+v} \\ &-\beta E_v(z-v+2a\rho) + D_v\frac{N(z,v)}{P(v)} + E_v\overline{\mathcal{I}}(z;-v). \end{aligned} \tag{114}$$

We choose D_v, E_v such that the first term is $P(v)\delta_{zv}$ and the second vanishes:

$$-\beta H(v)D_v + \frac{\lambda h^2}{\rho}E_v = P(v), \qquad \lambda\rho D_v - \beta H(-v)E_v = 0. \tag{115}$$

The determinant of this system is $\beta^2 H(v)H(-v) - \lambda^2 h^2 = \beta^2 e(v) \neq 0$, and the solution is

$$D_v = -\frac{H(-v)P(v)}{\beta e(v)}, \qquad E_v = -\frac{\lambda\rho F(v)}{\beta^2}. \tag{116}$$

With these values the remaining terms are $1/(\beta e(v))$ times the left side of the identity (A7),

$$\begin{aligned} &\lambda\rho(z-v+2a\rho)P(v) - H(-v)N(z,v) - \frac{\lambda\rho}{\beta}P(v)\overline{\mathcal{I}}(z;-v) \\ &\quad = -e(v)N_2(z) + \lambda\rho z[P(v) - e(v)] + \lambda W(z;v^2). \end{aligned} \tag{117}$$

Here $W(z;s) \coloneqq W_0(s) + z^2 W_2(s)$ with polynomials W_0, W_2 of degree one in s, given in Appendix A. Both sides of Eq. 117 are polynomials in z and v (Lemma 17). So the remaining terms are the polynomial

$$\frac{1}{\beta}\left[-N_2(z) + \lambda\rho z(F(v)-1) + \lambda\frac{W(z;v^2)}{e(v)}\right]. \tag{118}$$

Step 2 (the constant). For $y = 1$ the left side of Eq. 112 is

$$-\beta H(z) + \lambda\overline{c} + \mathrm{PV}\sum_l \frac{N(z,z_l)}{P(z_l)} = \frac{\beta N_2(z)}{2\varepsilon_r a}, \tag{119}$$

by $\mathcal{C}[1] = \overline{c}$ (Lemma 15) and identity (A6). So adding the constant $2\varepsilon_r a/\beta^2$ to y removes the first term of Eq. 118.

Step 3 (pole pairs). The rest of Eq. 118 must be removed by functions that are mapped to polynomials. A simple pole $1/(t-x)$ is mapped by $-\beta H + \lambda\,\mathcal{C}$ to poles at x and at $-x$ (Lemma 15). So consider $\varphi = f_+/(t-x) + f_-/(t+x)$. For it, the left side of Eq. 112 is a polynomial plus

$$\frac{-\beta H(x) f_+ + \lambda \alpha_k(-x) f_-}{z-x} + \frac{\lambda \alpha_k(x) f_+ - \beta H(-x) f_-}{z+x}. \tag{120}$$

For $x \neq 0$, both residues vanish for some $(f_+, f_-) \neq 0$ if and only if the determinant

$$\beta^2 H(x) H(-x) - \lambda^2 \alpha_k(x) \alpha_k(-x) = \beta^2 e(x) \tag{121}$$

vanishes (Eq. 98), i.e. if and only if $x = \pm u_j$. For $x = u_j$ we take

$$\varphi_j(t) := \frac{\beta H(-u_j)}{t-u_j} + \frac{\lambda \alpha_j}{t+u_j} = \frac{K_j t + u_j\big[\beta H(-u_j) - \lambda \alpha_j\big]}{t^2 - s_j}, \qquad \alpha_j := \alpha_k(u_j). \tag{122}$$

(For $u_j = 0$ the two poles coincide, $\varphi_j = K_j/t$, and what follows holds unchanged.) With $H(z) - H(\pm u_j) = (z \mp u_j)(z \pm u_j + 2a\rho)$, the polynomial part is

$$-\beta\big[K_j(z + 2a\rho) + u_j\big(\beta H(-u_j) - \lambda \alpha_j\big)\big] + \beta H(-u_j)\mathcal{I}(z; u_j) + \lambda \alpha_j \mathcal{I}(z; -u_j). \tag{123}$$

To evaluate it, set $v = \mp u_j$ in Eq. 117, where $e(v) = 0$ and $P(v) = P(u_j)$:

$$\frac{\lambda\rho}{\beta} P(u_j) \overline{\mathcal{I}}(z; \pm u_j) = \lambda\rho(\pm u_j + 2a\rho) P(u_j) - H(\pm u_j) N(z, \mp u_j) - \lambda W(z; s_j). \tag{124}$$

Insert this into Eq. 110 for $\mathcal{I}(z; \pm u_j)$, with $\alpha_k(-u_j) = h^2/\alpha_j$ (Eq. 98):

- The terms with $N(z, -u_j)$ carry the factor $\lambda^2 h^2 - \beta^2 H(u_j) H(-u_j) = -\beta^2 e(u_j) = 0$.
- The terms with $N(z, u_j)$ cancel.
- The terms with $\lambda\rho P(u_j)$ cancel the constant part of the first bracket.

What remains is

$$P(z)(\chi_r \varphi_j)(z) = -\beta K_j \left[z + \frac{W(z; s_j)}{\rho P(u_j)} \right]. \tag{125}$$

Step 4 (solving). P and e are monic and even of degree four, and $W(z; s)$ has degree one in s. Partial fractions in v^2 therefore give

$$F(v) - 1 = \sum_j F_j g_j(v), \qquad \frac{W(z; v^2)}{e(v)} = \sum_j \frac{W(z; s_j)}{s_j - s_\ell} g_j(v), \qquad F_j := \frac{P(u_j)}{s_j - s_\ell}. \tag{126}$$

So the last two terms of Eq. 118 are

$$\frac{1}{\beta} \sum_j \lambda\rho F_j g_j(v) \left[z + \frac{W(z; s_j)}{\rho P(u_j)} \right]. \tag{127}$$

By Steps 1–3 and Eq. 125,

$$y_v(z) := D_v \delta_{zv} + \frac{E_v}{z+v} + \frac{1}{\beta^2} \left[2\varepsilon_r a + \lambda\rho \sum_j \frac{F_j g_j(v)}{K_j} \varphi_j(z) \right] \tag{128}$$

solves Eq. 112. To read off Γ_r, note that by Eq. 122

$$\frac{\lambda\rho F_j g_j(v) \varphi_j(z)}{K_j} = \lambda\rho F_j z g_j(z) g_j(v) + \gamma_j g_j(z) g_j(v), \tag{129}$$

and that $F(z) - F(v) = -(z^2 - v^2) \sum_j F_j g_j(z) g_j(v)$ gives

$$\mathbb{D}(z, v) = \frac{F(v)}{z + v} - z \sum_j F_j g_j(z) g_j(v). \tag{130}$$

Hence

$$y_v(z) = -\frac{H(-z)P(z)}{\beta e(z)} \delta_{zv} + \frac{1}{\beta^2} \left[2\varepsilon_r a - \lambda\rho \mathbb{D}(z, v) + \sum_j \gamma_j g_j(z) g_j(v) \right]. \tag{131}$$

By Eq. 11 and Eq. 23, $B_n = -\beta G_2(z_n)/P(z_n)$; so by Eq. 16, $\beta^2 \left[y_{z_m}(z_n) - \delta_{nm}/B_n \right]$ is the right side of Eq. 40.

Step 5 (two-sidedness, branches, special cases).

- *Two-sidedness.* Let $Y_{nm} \coloneqq y_{z_m}(z_n)$, so that $\chi_r Y = \mathbb{1}$. Y is symmetric because $\mathbb{D}$ is, and so is χ_r (Eq. 18, Eq. 19, Eq. 24). Hence, with the same absolutely convergent series,

$$(Y\chi_r)_{nm} = \sum_l \chi_{r,ml} Y_{ln} = (\chi_r Y)_{mn} = \delta_{nm}. \tag{132}$$

- *Branches.* Let $K(u) \coloneqq \beta H(-u) + \lambda \alpha_k(u)$. Since $\alpha_k(u)\alpha_k(-u) = h^2$ (Eq. 98), for every u

$$\frac{\beta H(-u) - \lambda \alpha_k(u)}{K(u)} + \frac{\beta H(u) - \lambda \alpha_k(-u)}{K(-u)} = \frac{2\beta^2 e(u)}{K(u)K(-u)}. \tag{133}$$

  At a root the right side vanishes, so Z_j and γ_j do not depend on the sign of u_j.
- *Confluent roots (item 2 of Proposition 6).* By Eq. 133, the odd part

$$\frac{Z(u) - Z(-u)}{2} = \frac{\beta^2 u P(u) e(u)}{K(u)K(-u)} \tag{134}$$

  vanishes at the roots, so $Z(s_j) = Z_j$ for the even part $Z(s)$. Hence the sum over j in Eq. 40 is $\lambda\rho$ times the divided difference of $s \mapsto Z(s)/((z_n^2 - s)(z_m^2 - s))$ at s_1, s_2. Near a double root $u_\star^2$, $Z(u)$ is analytic in u at $\pm u_\star$: $K(u_\star) \neq 0$, and at a root $K(u) = 0$ if and only if $K(-u) = 0$ (Eq. 98, Eq. 36). So $Z(s)$, an even analytic function of u, is analytic in s, also for $u_\star = 0$. As μ approaches a value where $s_1 = s_2$ (a zero of the discriminant of e, a nonconstant analytic function of μ), the divided difference converges to the derivative, and Lemma 18 extends $\chi_r Y = \mathbb{1}$.
- *The atomic pair.* For $r = \mathrm{m}$ and $k = 0$ the atomic pair has $F_1 = P(a)/(a^2 - s_2) = 0$ and $W(z; a^2) = 0$ (Appendix A). So its terms in Step 4 vanish, and Step 3 is needed only for the non-atomic pair.
- *Excluded cases.* Steps 1–4 assumed $P(u_j) \neq 0$ for the non-atomic pairs; they hold for $u_j = 0$ as well. For $r = \mathrm{m}$, $k = 0$, $P(u_2) = 0$ would mean $s_2 = a^2 = s_1$, which is excluded. Otherwise, fix k: the values of μ where the assumption fails are zeros of the resultant of e and P, a nonconstant analytic function of μ, so such values are isolated. Eq. 40 has no $P(u_j)$ in a denominator and depends on s_j only. Near such a value, $\chi_{r,nl} = O(|z_l|^{-2})$ for $l \neq n$ and Y_{lm} is bounded, uniformly; so Lemma 18 extends $\chi_r Y = \mathbb{1}$.
- *Lattice roots.* The case $e(z_n) = 0$ (item 3 of Proposition 6) is treated in Appendix B. □

Acknowledgment. The derivations, proofs and verification scripts were developed with the help of AI assistants (Anthropic's Claude and OpenAI's GPT models). Every identity is checked by computer algebra (Appendix A) and every result numerically (Appendix D); Appendix E lists what is formalized in Lean.

Bibliography

[1] P. Thunström, O. Gunnarsson, S. Ciuchi, and G. Rohringer, Analytical investigation of singularities in two-particle irreducible vertex functions of the Hubbard atom, Physical Review B **98**, 235107 (2018).

[2] H. Eßl, M. Reitner, G. Sangiovanni, and A. Toschi, General Shiba mapping for on-site four-point correlation functions, Physical Review Research **6**, 033061 (2024).

[3] S. Rohshap, Analytical Calculation of Two-Particle Vertices for Multiorbital Hubbard Atom, Master's thesis, 2023.

[4] H. Eßl, Breakdown of the Self-Consistent Perturbation Theory Beyond Particle-Hole Symmetry, Master's thesis, 2023.

A Rational identities

Each identity is checked exactly with SymPy: (A1)–(A5) by `doped-gamma/verify_identities.py`, (A6) and (A7) by `doped-gamma/verify_dm.py` (both channels, $k \neq 0$ and $k = 0$, all parameters free). Lean proves them too (Table 1).

(A1) the twelve terms of Appendix C equal the two-term form in the proof of Lemma 1; and $\Pi(x_n, x_m) = (z + v - 2a)^2/(Q_-(z)Q_-(v))$.

(A2) the z^0 coefficient in the proof of Proposition 16:

$$\begin{aligned} &4a^2\rho(2a - v)Q_+(v) - 2aq_0Q_+(v) + 4a^2(1 - \rho)q_0(v + 2a) \\ &= -2a(v^2 + 2q_0 - 4a^2)(v^2 + 2a\rho v + q_0) + 2aP(v). \end{aligned} \tag{135}$$

(A3) Eq. 63; $zv + q_0 = a^2 + (t_n + t_m)/2$ at transfer $w' = b - x_n - x_m$; Eq. 69 and Eq. 70 from the twelve terms of Appendix C at w' with $c = \rho/b$.

(A4) $\Xi = \mathrm{diag}(-1/a, K_\mathrm{s}/(ad_+d_-))$ in the proof of Theorem 3, and $d_+d_- = \mathcal{P}(-A_\mathrm{s})$.

(A5) the δ_{nm} and δ_{k0} identities in the proof of Proposition 16, and $H(a) = 4a^2p_2$, $H(-a) = 4a^2p_0$ for $r = \mathrm{m}$, $k = 0$.

(A6) Step 2 of the proof of Theorem 4, the sum evaluated by Lemma 9 and Lemma 17:

$$-\beta H(z) + \lambda\overline{c} + \mathrm{PV}\sum_l \frac{N(z, z_l)}{P(z_l)} = \frac{\beta N_2(z)}{2\varepsilon_r a}. \tag{136}$$

(A7) Eq. 117, where $\overline{\mathcal{I}}$ is evaluated by Lemma 17 and, with $\varpi \coloneqq 4(p_0p_2 - p_1^2)$,

$$\begin{aligned} W_0(s) &\coloneqq \varepsilon_r a q_0[\varpi - 2\varepsilon_r(1 - \rho^2)](s + q_0 + 2\varepsilon_r a^2) + \delta\beta a^4\psi(s), \\ W_2(s) &\coloneqq \varepsilon_r a\varpi(s + q_0 + 2\varepsilon_r a^2) - \delta\beta a^2\psi(s), \\ \psi(s) &\coloneqq -4\varepsilon_r p_1(p_0 + p_2 - \rho^2)[s + (1 + 2\varepsilon_r)a^2] + (1 + \varepsilon_r)\rho^2\varpi(s - a^2). \end{aligned} \tag{137}$$

For $r = \mathrm{m}$ and $k = 0$ ($\varepsilon_\mathrm{m} = -1$, $q_0 = a^2$) this shows $W_0(a^2) = W_2(a^2) = 0$.

B Analytic details

Lemma 18 (continuity). Let $\chi(p)$ and $Y(p)$ be matrices indexed by the lattice, depending on a real parameter p (μ or β) near p_0, with entries continuous at p_0. Suppose that for all n, m there are c_{nm} and a finite set of indices l outside of which, for all these p,

$$|\chi(p)_{nl}Y(p)_{lm}| \le \frac{c_{nm}}{1+l^2}. \tag{138}$$

If $\chi(p)Y(p) = \mathbb{1}$ for $p \neq p_0$, then also $\chi(p_0)Y(p_0) = \mathbb{1}$.

Proof. For fixed n, m, the finitely many exceptional terms of $\sum_l \chi_{nl}Y_{lm}$ are continuous at p_0, and the sum of the others is continuous by dominated convergence. □

Lattice zeros at $\mu \neq 0$ (item 3 of Proposition 6). Since e has real coefficients and $z_{\overline{n}} = z_n^*$, $e(z_{\overline{n}}) = e(z_n)^*$: lattice zeros of e come in mirror orbits $O = \{n, \overline{n}\}$. Let e vanish on one orbit O at $\mu = \mu_0 \neq 0$, the other assumptions of Theorem 4 holding at μ_0. As $e(z_n)$ is a nonconstant analytic function of μ, Theorem 4 holds for $\mu \neq \mu_0$ near μ_0. We show that the entries of its inverse Y (Step 4) converge as $\mu \to \mu_0$, boundedly off the diagonal; then Lemma 18 gives $\chi_r Y = \mathbb{1}$ at μ_0.

- *Roots near the lattice.* Each root u_j that tends to a point z_c, $c \in O$, has $(z_c - u_j)\alpha_j \to \rho \neq 0$ and $K_j \to \infty$, since $(z_c - u)S_k(u) \to 1$ as $u \to z_c$ (Lemma 8).
- *Columns off the orbit.* In the column y_v of Step 4 with $v = z_m$, $m \notin O$, the coefficients D_v, E_v and $F_j g_j(v)$ stay bounded. By Eq. 122, for $c \in O$,

$$\frac{\varphi_j(z_c)}{K_j} = \frac{\beta H(-u_j)/(z_c - u_j) + \lambda\alpha_j/(z_c + u_j)}{\beta H(-u_j) + \lambda\alpha_j} \to \frac{\beta H(-z_c)}{\lambda\rho} + \frac{1}{2z_c} \quad (u_j \to z_c); \tag{139}$$

  it tends to $1/(z_c + z_{c'})$ if $u_j \to z_{c'}$ with $c' \neq c$, and it stays bounded otherwise. So $Y_{cm} = y_{z_m}(z_c) = Y_{mc}$ converges for $c \in O$, $m \notin O$, boundedly in m.
- *The block on the orbit.* Row $l \notin O$ of $\chi_r Y = \mathbb{1}$ gives

$$\sum_{c\in O} \chi_{r,lc}Y_{cc'} = -\sum_{l'\notin O} \chi_{r,ll'}Y_{l'c'}, \tag{140}$$

  whose right side converges by dominated convergence. By Eq. 104, the columns

$$P(z_l)\chi_{r,lc} = \frac{\lambda\rho}{z_l + z_c} + \frac{N(z_l, z_c)}{P(z_c)}, \quad c \in O, \tag{141}$$

  are linearly independent rational functions of z_l, with distinct poles at $-z_c$. So $|O|$ rows $l \notin O$ can be chosen for which the matrix $(\chi_{r,lc}(\mu_0))$ is invertible, and the block converges.

Lattice zeros at $\mu = 0$ (item 4 of Proposition 6). Let $\beta H(z_n) = \lambda h$ with $n \neq \overline{n}$ at $\beta = \beta_0$, so that $z_n^2 = \hat{s}$ and

$$e(z) = (z^2 - \hat{s})(z^2 - \hat{s} + 2\lambda h/\beta). \tag{142}$$

As $\beta \to \beta_0$, $\hat{u} \to z_n$. The diagonal and mirror terms of Eq. 44 contribute

$$-\frac{\lambda h P(z_n)}{e(z_n)} + O(1) = -\frac{\beta P(z_n)}{2(z_n^2 - \hat{s})} + O(1) \tag{143}$$

to each of the entries (n, n) and $(n, \overline{n})$. Since $S_k(\hat{u}) = 1/(z_n - \hat{u}) + O(1)$,

$$\hat{D} = \frac{2\varepsilon_r a}{\hat{u}(\hat{u} - z_n)} + O(1), \tag{144}$$

so the separable term contributes $\beta P(z_n)/(2(z_n^2-\hat{s}))+O(1)$ to the same entries and $O(1)$ to the others. The singular parts cancel.

At a mirror fixed point $n=\overline{n}$ (odd k, $z_n=0$), let $e(0)=0$ at $\beta=\beta_0$. The diagonal and mirror terms of Eq. 44 act on the same entry (n,n). As $\beta^2 e(0)=(\beta H(0)-\lambda h)(\beta H(0)+\lambda h)$, they combine to

$$\frac{\beta P(0)}{G_2(0)}-\frac{\beta^2 P(0)}{\beta H(0)-\lambda h}. \tag{145}$$

- If $\beta H(0)=-\lambda h$ at β_0, this stays finite, and under the remaining assumptions of Corollary 5 nothing else is singular.
- If $\beta H(0)=\lambda h$, then $\hat{s}\to 0$, and the combination is $\beta P(0)/\hat{s}+O(1)$, as $\beta H(0)-\lambda h=-\beta\hat{s}$. For odd k, $\coth(\beta\hat{u}/2)/\hat{u}=2/(\beta\hat{s})+O(1)$, so $\hat{D}=2\varepsilon_r a/\hat{s}+O(1)$, and the separable term is

$$-\frac{2\varepsilon_r a\beta P(\hat{u})}{\hat{D}}\hat{g}(0)^2=-\frac{\beta P(0)}{\hat{s}}+O(1) \tag{146}$$

  at (n,n) and $O(1)$ elsewhere. The singular parts cancel again.

In all these cases Eq. 44 extends continuously, and Lemma 18 (with $p=\beta$) gives $\chi_r Y=\mathbb{1}$ at β_0.

Proof of Corollary 5. Take $\mu\to 0$ at fixed k in Eq. 40. Then $\rho\to 0$, $\overline{c}\to h$ and $C_{nm}\to h\delta_{m\overline{n}}$ (Eq. 21). One root s_j of e tends to $\hat{s}$, the other, s_ℓ, to $\hat{s}'$. At a root u_j put $X_\pm := \beta H\big(\pm u_j\big)$. Then, by Eq. 36,

$$X_+X_-=\lambda^2h^2,\qquad X_+-X_-=4a\beta\rho u_j, \tag{147}$$

and with $\overline{c}^2=h^2+\beta^2\rho^2/4$ (Eq. 6) this gives

$$(X_--\lambda\overline{c})(X_-+\lambda\overline{c})=-\rho\big(4a\beta u_jX_-+\lambda^2\beta^2\rho/4\big). \tag{148}$$

By Eq. 38,

$$K_j=X_--\lambda\overline{c}+\lambda\rho S_k\big(u_j\big),\qquad X_--\lambda\alpha_k\big(u_j\big)=X_-+\lambda\overline{c}-\lambda\rho S_k\big(u_j\big). \tag{149}$$

- *The pair j.* Here $X_\pm\to\lambda h$, so the identity gives $(X_--\lambda\overline{c})/\rho\to -2a\beta\hat{u}$. Hence

$$\frac{K_j}{\rho}\to\lambda S_k(\hat{u})-2a\beta\hat{u}=-2a\hat{u}\hat{D},\qquad X_--\lambda\alpha_k\big(u_j\big)\to 2\lambda h,\qquad s_j-s_\ell\to\frac{2\lambda h}{\beta}, \tag{150}$$

  so $\gamma_j\to -2\varepsilon_r a\beta P(\hat{u})/\hat{D}$ if $\hat{u}\neq 0$.
- *The pair ℓ.* Here $X_\pm\to-\lambda h$, so $K_\ell\to-2\lambda h$, $X_--\lambda\alpha_k(u_\ell)\to 0$ and $\gamma_\ell\to 0$. (For $r=\mathrm{m}$, $k=0$ this is the atomic pair, absent anyway.)
- *The dense term.* By Eq. 130, since $z_{\overline{n}}=-z_n$ at $\mu=0$ and F is even,

$$\lambda\rho\mathbb{D}(z_n,z_m)=\lambda C_{nm}F(z_m)-\lambda\rho z_n\sum_j F_jg_j(z_n)g_j(z_m)\to\lambda hF(z_n)\delta_{m\overline{n}}. \tag{151}$$

So Eq. 40 tends entrywise to Eq. 44.

It remains to show that the limit is an inverse. For small $\mu\neq 0$ the assumptions of Theorem 4 hold: $e(z_n)\neq 0$ by continuity (automatically for large $|n|$), $s_1\neq s_2$, $K_\ell\neq 0$, and $K_j\neq 0$ because $K_j/\rho\to -2a\hat{u}\hat{D}\neq 0$ (for $\hat{u}\neq 0$). The entries of C are bounded uniformly near $\mu=0$ (by $\beta\rho/(2\pi)$ for $m\neq\overline{n}$, and $\rho/(2\mu)\to\beta p_0$). So $\chi_{r,nl}=O(|z_l|^{-2})$ for $l\neq n$, and the entries of $Y=\Gamma_r/\beta^2+\boldsymbol{B}^{-1}$, with Γ_r from Eq. 40, are bounded off the diagonal, uniformly near $\mu=0$. So Lemma 18 gives $\chi_rY=\mathbb{1}$ at $\mu=0$; Y stays symmetric. Finally, for even k, $\hat{u}=0$ occurs at isolated β, where Eq. 44 is continuous in β (as $\hat{D}\neq 0$); Lemma 18 with $p=\beta$ covers it. For odd k, $\hat{u}=0$ means $e(0)=0$ at the mirror fixed point, treated above. □

Proof of Proposition 7. Rescale χ_r to an operator on ℓ^2,

$$\hat{\chi} \coloneqq \boldsymbol{\vartheta}\chi_r\boldsymbol{\vartheta}, \qquad \boldsymbol{\vartheta} \coloneqq \operatorname{diag}(\vartheta_n), \qquad \vartheta_n \coloneqq \left(1 + |z_n - \mu|^2\right)^{1/2}. \tag{152}$$

- $\hat{\chi} = \kappa_r \mathbb{1} + \hat{\chi}_c$ *with* $\hat{\chi}_c$ *compact*, where $\kappa_\mathrm{d} = \kappa_\mathrm{m} = \beta$ and $\kappa_\mathrm{s} = -\kappa_\mathrm{t} = \beta/2$.
  - $r = \mathrm{d}, \mathrm{m}$: by Eq. 104, $\chi_{r,nn} = -\beta H(z_n)/P(z_n) + O(z_n^{-4})$, so $\vartheta_n^2 \chi_{r,nn} \to \beta$. Off the diagonal, $\chi_{r,nm} = O(|z_n|^{-2}\,|z_m|^{-2})$ by Eq. 18, Eq. 19 and Eq. 24. So these entries of $\hat{\chi}$ are square-summable, i.e. Hilbert–Schmidt.
  - $r = \mathrm{s}, \mathrm{t}$: $t_n/\vartheta_n^2 \to 1$, so by Eq. 64

$$\vartheta_n^2 B_n \to -\frac{\beta}{2}, \qquad \vartheta_n^2 L_\mathrm{s}(t_n) \to \frac{\beta}{2}, \qquad \vartheta_n^2 L_\mathrm{t}(t_n) \to -\frac{\beta}{2}. \tag{153}$$

    By Eq. 68, $\hat{\chi} - \kappa_r\mathbb{1}$ is a matrix supported on the diagonal and the mirror with entries tending to zero, which is compact, plus (for $r = \mathrm{s}$) $\boldsymbol{\vartheta}\chi_{\mathrm{s},\,\mathrm{reg}}\boldsymbol{\vartheta}$, which is Hilbert–Schmidt by Eq. 69.
- $\hat{\chi}$ *is invertible.* For each m, $\hat{\chi}$ maps $\boldsymbol{\vartheta}^{-1}\chi_r^{-1}\boldsymbol{e}_m$ to $\vartheta_m \boldsymbol{e}_m$: the n-th entry of the image is $\vartheta_n \sum_l \chi_{r,nl}(\chi_r^{-1})_{lm} = \vartheta_n \delta_{nm}$, an absolutely convergent series. The vector $\boldsymbol{\vartheta}^{-1}\chi_r^{-1}\boldsymbol{e}_m$ lies in ℓ^2, because by the explicit formulas $(\chi_r^{-1})_{nm}$ is bounded in n for $n \notin \{m, \overline{m}\}$. So the range of $\hat{\chi}$ contains all basis vectors; being closed (a compact perturbation of $\kappa_r\mathbb{1}$ has closed range), it is all of ℓ^2. A compact perturbation of $\kappa_r\mathbb{1}$ that is onto is also injective (Fredholm alternative), hence invertible.
- *Convergence.* Let $\mathbb{1}_M$ be the projection onto the box. Then

$$\hat{\chi}_M \coloneqq \mathbb{1}_M \hat{\chi} \mathbb{1}_M + \kappa_r(\mathbb{1} - \mathbb{1}_M) \tag{154}$$

  converges to $\hat{\chi}$ in norm, since $(\mathbb{1} - \mathbb{1}_M)\hat{\chi}_c \to 0$ and $\hat{\chi}_c(\mathbb{1} - \mathbb{1}_M) \to 0$ for compact $\hat{\chi}_c$. So $\hat{\chi}_M$ is invertible for all sufficiently large M, and $\hat{\chi}_M^{-1} \to \hat{\chi}^{-1}$ in norm. On the box, $\hat{\chi}_M^{-1} = \boldsymbol{\vartheta}^{-1}(\mathbb{1}_M \chi_r \mathbb{1}_M)^{-1}\boldsymbol{\vartheta}^{-1}$.

□

C The regular part

The regular part in Eq. 20 is [Ref. [2], Eq. (D3), at zero field]

$$\begin{aligned} R_{\mathrm{reg}}(x,y) = (p_0+p_1)\Big[&-\frac{1}{(y+w+a)(y+a)(x-a)}+\frac{1}{(x+w-a)(x-a)(y+a)} \\ &+\frac{1}{(x+w+a)(y-a)(x-a)}-\frac{1}{(x+w+a)(x-a)(y+w+a)} \\ &+\frac{1}{(y+a)(x+w+a)(y+w+a)}+\frac{1}{(y+a)(x+w+a)(x+a)} \\ &+\frac{1}{(y-a)(x+a)(y+w-a)}-\frac{1}{(y-a)(x+w+a)(x+a)} \\ &+\frac{1}{(x+w-a)(x+a)(y+w-a)}-\frac{1}{(y+a)(x+w-a)(x+a)}\Big] \\ -(p_1+p_2)\Big[&\frac{1}{(y-a)(x+w-a)(y+w-a)}+\frac{1}{(y-a)(x+w-a)(x-a)}\Big]. \end{aligned} \tag{155}$$

D Numerical checks

- **Mathematica:** `mathematica/doped_gamma.nb` derives Eqs. 18–20 exactly from the Lehmann representation of Eq. 3, checks Lemma 1, Proposition 11, Proposition 16, the lemmas of Section 5 and Section 7, Theorem 2, Theorem 3, Steps 1–4 of the proof of Theorem 4 and (A2)–(A4), (A6), (A7) symbolically, and compares Theorem 4, Theorem 3, Theorem 2 and Corollary 5 with finite-box inversion (62 checks).
- **Vertices** (`doped-gamma/check_vertices.jl`): Eq. 32 and Eq. 34 agree with an independent residue-based BigFloat evaluator (`doped-gamma/doped_gamma.jl`) to 3×10^{-15}, Eq. 40 at 256 bits to 10^{-65} (300 entries, $k=0,1,2,-3,5$, $|\mu|\le 2\,|U|$); at $\mu=0$, Eq. 32, Eq. 34 and Eq. 44 agree with `HubbardAtoms.jl` (Eq. 19 of Ref. [1]) to 3×10^{-14}.
- **Expanded formulas** (`doped-gamma/check_explicit.jl`): Appendix F, in the notation of Ref. [1], is transcribed without the main text's polynomial and thermal coefficients and compared with `DopedHubbardAtom` at 256 bits, including diagonal entries, mirror fixed points, both signs of each particle–hole square root and the static magnetic atomic root.
- **Field sectors** (`doped-gamma/check_field.jl`): the displayed field formulas of Appendix F.6 against `FieldHubbardAtom` at 256 bits; divergence conditions, growing susceptibility boxes, null-vector overlaps and branch cancellations (Appendix G.5.5).
- **Divergences** (`doped-gamma/check_divergences.jl`):
  - On 30 cuts at fixed $\beta\mu\in[0.05,3.9]$, $|\beta U|\le 25$, each of the 70 lines of Eq. 212 and Eq. 204 matches a change by one of the number of negative real eigenvalues of χ_{d}, χ_{m}, χ_{s} on the box $|2n+1|\le 49$ at an unchanged number of conjugate pairs. The scan steps βU by 0.05 and matches within 0.15. There are no other such changes, and none by more than one. Changes of the number of conjugate pairs (pairs turning real, pseudo-divergences) are not compared.
  - At 8 points on the lines (including $\mu=0$), the box eigenvector of the eigenvalue closest to zero has overlap $|\cos\angle(\boldsymbol{v},\boldsymbol{g})|$ with the $\boldsymbol{g}$ of Appendix G that differs from 1 by less than 2×10^{-12}.
  - At the three exceptional points quoted in Appendix G, on the boxes $|2n+1|\le 121$ and 241, exactly two eigenvalues have eigenvectors aligned with $\boldsymbol{g}_j$ (overlap above 0.99). They agree in size within 10%, are below a tenth of all other eigenvalues, and shrink by more than half from the smaller box to the larger (10^{-8} on the larger). An ordinary divergence and a point off the lines fail this test.

- On the singlet curve $K_\mathrm{s} = 0 = d_\mp$ at $\beta\mu = \pm 0.3, \pm 1, \pm 3$, the box eigenvalue whose eigenvector is most aligned with $1/(t_n + A_\mathrm{s})$ stays above 0.02, and Γ_s (`DopedHubbardAtom`) converges to the same finite value from both sides of the curve.
- The conditions of Appendix G.4, transcribed with $B^2 = -s_j$, vanish on six of the lines to 10^{-13}, and its sum for the exceptional points, summed directly over ν, vanishes at two of them to 10^{-10} of the sum of the moduli.
- A point where two arcs of $\Phi(s_j) = 0$ cross (equality in the sign condition of Eq. 212) is a divergence: the smallest box eigenvalue shrinks from 10^{-10} to 3×10^{-14}.

- **Implementation:** the test suite of `HubbardAtoms.jl` compares `DopedHubbardAtom` with exact diagonalization, with finite-box inversion and with `BigFloat` evaluation, including the removable singularities of Proposition 6.

E Formalization in Lean

The Lean 4 project `lean/`, next to this note in the repository of HubbardAtoms.jl, builds on Mathlib. It derives the input of Section 3 from the Hamiltonian, and formalizes Theorem 2, Theorem 3, Theorem 4, Corollary 5 and most of Proposition 6, together with the statements their proofs use. Table 1 lists each numbered statement with what is formalized and where. The names are declarations in the namespace `HubbardAtom.Params`, the auxiliary rational identities in `HubbardAtom`; the versions of Lean and Mathlib are pinned in the project. Its script `scripts/check.sh` builds it from scratch, fails on any warning (an unfinished proof, `sorry`, is one), and checks that every declaration depends only on Lean's three standard axioms.

Taken as given. The theorems are statements about definitions, which the reader has to compare with this note:

- the model of Section 2: Eq. 3 on the four states, with $c_\uparrow$, $c_\downarrow$ as 4×4 matrices (Lean checks their anticommutation relations), $O(\tau) = e^{\tau\mathcal{H}} O e^{-\tau\mathcal{H}}$, the thermal average $\mathrm{Tr}\big(e^{-\beta\mathcal{H}} X\big) / \mathrm{Tr}\, e^{-\beta\mathcal{H}}$, the fermionic time ordering, and the integrals defining G and $\chi_{\sigma\sigma'}$; the frequencies and the lattice of Section 2.4. From these, Lean derives the weights, Eq. 11 and the input Eqs. 18–20 with Appendix C, for all U, μ and n, m, k;
- the particle–particle form, the channel combinations, Eq. 15 and Eq. 16. A right inverse Y of χ means $\sum_l \chi_{nl} Y_{lm} = \delta_{nm}$ for all n, m, with unconditionally, i.e. absolutely, convergent sums; likewise for left inverses;
- the closed forms Eq. 32, Eq. 34, Eq. 46, Eq. 40 with its limit in item 2 of Proposition 6 (where $\partial_s Z$ is written out and shown to be the s-derivative of $Z(s)$), Eq. 44 (in a form combining the diagonal and mirror terms, equal to Eq. 44 where $e(z_n) \neq 0$) and Eq. 49, with $\mathbb{D}(z, z)$ through the derivative of F, and the assumptions, as typed in Lean.

The inverses constructed in the proofs need no trust: Lean proves that they are inverses from both sides.

Statement	In Lean	Declarations
Input, Eqs. 18–20, Eq. 11, Appendix C	derived from Eq. 3 for all U, μ, n, m, k	`propagator_eq`, `chiLehmann_upUp`, `chiLehmann_upDn` (`LehmannInput.lean`)
Lemma 1 (compact input)	for $\mu \neq 0$	`compact_input` (`PHInput.lean`)
Theorem 2 (triplet)	as stated; moreover every right or left inverse of χ_t gives Eq. 32, so Γ_t is unique	`exists_inverse_chiT`, `gammaT_of_isRightInverse`, `gammaT_of_isLeftInverse`, `gammaT_eq_zero_of_fixed` (`Triplet.lean`)
Theorem 3 (singlet)	as stated, for all b (including $b = 0, \pm 2a$, $d_\pm = 0$, $\xi = 0$): an inverse of χ_s from both sides, and Eq. 34; for $U = 0$, $\Gamma_\mathrm{s} = 0$ for every inverse	`SingletHyp.chiS_isRightInverse`, `SingletHyp.chiS_isLeftInverse`, `SingletHyp.gammaS_eq` (`SingletZero.lean`); `gammaS_zero_of_a_eq_zero` (`UZero.lean`)

Statement	In Lean	Declarations
Theorem 4 (density and magnetic)	as stated: an inverse of χ_r from both sides and Eq. 40, including $\mathbb{D}(z,z)$, the case $r = \mathrm{m}$, $k = 0$, and the independence of the sign of u_j; for $U = 0$, $\Gamma_r = 0$ for every inverse	`DMHyp.toDMGeneric`, `DMGeneric.chiR_isRightInverse`, `DMGeneric.chiR_isLeftInverse`, `DMGeneric.gammaR_eq`, `DMGeneric.gammaR_neg₁`, `dmGeneric_static_magnetic` (`DensityMagnetic.lean`, `DMLifts.lean`); `gammaR_zero_of_a_eq_zero` (`UZero.lean`)
Corollary 5 (half filling)	for $U \neq 0$, directly at $\mu = 0$ rather than as a limit, including $\hat{u} = 0$, assuming instead of $e(z_n) \neq 0$ only $z_n^2 \neq \hat{s}$ for all n and $z_n^2 \neq \hat{s}'$ off the mirror fixed points; so it also holds at a fixed point with $\beta H(0) = -\lambda h$	`HFHyp.of_eph`, `HFHyp.chiR_isRightInverse`, `HFHyp.chiR_isLeftInverse`, `HFHyp.gammaHF_eq`, `gammaHF_eq_note` (`HalfFilling.lean`)
Proposition 6 (removable singularities)	items 1 and 4: the values Eq. 46 (for all b) and Eq. 49 (including $\hat{s} = 0$ at a fixed point) give inverses from both sides; that they are the limits is not formalized. Item 2: as stated, including $s_1 = s_2 = 0$. Item 3: not formalized	item 1: `SingletLZ.chiS_isRightInverse`, `SingletLZ.chiS_isLeftInverse`, `SingletLZ.gammaS_eq` (`SingletLZ.lean`); item 2: `DMConfHyp.toDMConf`, `DMConf.chiR_isRightInverse`, `DMConf.chiR_isLeftInverse`, `DMConf.gammaR_eq`, `DMConf.tendsto_Zeven` (`Confluent.lean`); item 4: `HFLZ.chiR_isRightInverse`, `HFLZ.chiR_isLeftInverse`, `HFLZ.gammaHF_eq` (`HalfFillingLZ.lean`)
Proposition 7 (finite boxes)	not formalized	
Lemma 8 (basic sum)	for $j = 1$ along the boxes $\lvert 2n + 1 + k\rvert \leq 2M + 1$, and for $j = 2$, from Mathlib's expansion of the cotangent; Eq. 53 and $S_{\pm}$	`hasSum_S_even`, `hasSum_S_odd`, `hasSum_sq` (`Matsubara.lean`); `S_shift`, `S₀_neg_a`, `S₀_a` (`Thermal.lean`); `S_shift'` (`PHSums.lean`)
Lemma 9 (residue rule)	not as such: the sums it serves, with the denominators $P(t)$ and $P(t)(t - x)$, are evaluated by splitting $P = Q_+ Q_-$, for all k at once	`hasSum_Q_sub`, `hasSum_inv_Q`, `hasSum_cubic` (`PHSums.lean`)
Lemma 10 (particle–particle sum)	as stated, including $\xi = 0$	`hasSum_Tpp`, `hasSum_one_div_t_sub` (`Matsubara.lean`); `hasSum_gp`, `hasSum_gm`, `Tpp_tp`, `Tpp_tm` (`Singlet.lean`)
Proposition 11 (pp)	as stated, with $\chi_{\mathrm{s,\,reg}}$ in the form Eq. 70 for $b \neq 0$, and as the partial fractions of Eq. 69 for $b = 0$	`chiT_eq_mirrorBlock` (`Triplet.lean`); `chiS_eq` (`SingletInput.lean`); `chiS_eq_zero` (`SingletZero.lean`)
Lemma 12 (values on the reflected lattice)	the last identity; the value of S'_k enters Lemma 13	`rho_mul_S_neg_z` (`PHSums.lean`)
Lemma 13 (square)	as stated	`hasSum_square` (`PHSums.lean`)
Lemma 14 (thermal function)	as stated	`αeven_eq`, `αodd_eq` (`Thermal.lean`); `α_mul_α_neg` (`PHSums.lean`)
Lemma 15 (Cauchy calculus)	for simple poles; the constant rule only together with the sum of Lemma 17. Higher poles are not needed	`hasSum_cauchy`, `hasSum_const` (`PHSums.lean`)
Proposition 16 (structure)	as stated, for $\mu \neq 0$ and at $\mu = 0$, where $C = h\mathcal{M}$	`Pph_mul_chiR` (`PHInput.lean`); `Pph_mul_chiR_hf` (`HalfFilling.lean`)
Lemma 17 (rank-two sums)	as stated, with the residues at the zeros of P written out	`hasSum_rank_two`, `Rhat` (`PHSums.lean`)
Proof of Theorem 4	Steps 1–4 with Eq. 130; of Step 5 the two-sidedness, Eq. 133 and the atomic pair	`DMGeneric.hasSum_step1`, `DMGeneric.hasSum_step2`, `DMGeneric.hasSum_step3`, `step4_identity`, `DMGeneric.DD_eq`, `chiR_symm`, `DMGeneric.invR_symm`, `atomic_root` (`DensityMagnetic.lean`); `Zph_sub_Zph_neg` (`Confluent.lean`)
Lemma 18 (continuity)	not formalized; Lean does without it (see below)	
(A1)–(A7) (Appendix A)	all; (A2) and the first part of (A5) within `Pph_mul_chiR`, and the consequences $e(a) = 0$, $W_0(a^2) = W_2(a^2) = 0$ for $r = \mathrm{m}$, $k = 0$ in `atomic_root`	(A1) `compact_input`, (A3) `regular_pp`, (A4) `singlet_A4`, (A6) `identity_A6`, (A7) `identity_A7`

Table 1: What the Lean project proves, statement by statement (files in `lean/HubbardAtomLean/`).

Proofs. For the input, Lean splits the integral over $[0,\beta]^3$ into the six orderings of the times, sums each ordering over the paths through the four states, and evaluates the integrals over the ordered simplex in closed form, also where a partial sum of the rates vanishes: at $k=0$, at $n=m$, and at $\mu=0$, $n+m+k+1=0$ (`chiLehmann_eq`, `orderedTerm_eq`, `pathTerm_eq`). Two arguments differ from this note. Lattice sums with the denominator P are evaluated by splitting $P=Q_+Q_-$ instead of by Lemma 9, which needs no separate treatment of the double zeros at $k=0$. And the inverse of χ_s is verified directly; the cancellation of the Woodbury step is `woodbury_identity`. Where this note takes limits, Lean argues directly. Corollary 5 is proved at $\mu=0$. At the lattice zeros of items 1 and 4 of Proposition 6 the limiting inverses are verified with lattice sums from which the orbit is removed (`SingletLZ.hasSum_g_kappa`, `HFLZ.hasSum_Nph_ghat`). For $s_1=s_2$, Step 3 holds for every u up to a remainder proportional to $e(u)$ (`step3_general`), which vanishes to second order at a double root; the image of the u-derivative of the mode then follows by differentiating the lattice sums termwise (`hasSum_deriv_of_bound`, from Mathlib's `Complex.hasSum_deriv_of_summable_norm`), and at $s=0$ from difference quotients by Tannery's theorem (`DMConf.hasSum_M₁_zero`).

Not formalized. Appendix G; Proposition 7, Lemma 18 and the rest of Appendix B: item 3 of Proposition 6 (the lattice zeros of e for $\mu\neq 0$), and that the values in items 1 and 4 are the limits of Eq. 34 and Eq. 44; that the inverses of χ_s, χ_d and χ_m are unique (for χ_t it is proved, and for $U=0$ in all channels); and floating-point evaluation.

F The vertices written out

This appendix gives the regular formulas of Theorem 2, Theorem 3, Theorem 4 and Corollary 5 without the nested coefficient definitions used in the proofs, in the notation of Ref. [1]. The frequencies are written directly: $\Gamma_r^{\nu\nu'\omega} \coloneqq \Gamma_r(\nu,\nu',\omega)$ in the channel convention of Eq. 16, which is that of Ref. [1]. At $\mu = 0$ the formulas are Eq. (19) of that reference. Besides the input panel, each channel defines a few constants, the counterparts of A_r and B_r in Ref. [1]; every polynomial and thermal coefficient is written out inside the vertex. Appendix G.4 relates this notation to that of the main text. These are exact identities, not numerically stabilized evaluation rules.

F.1 Input panel

Take $U, \mu \in \mathbb{R}$, $\beta > 0$, and the Matsubara frequencies $\nu = (2n+1)\pi/\beta$, $\nu' = (2m+1)\pi/\beta$, $\omega = 2\pi k/\beta$. For $U = 0$, all four vertices vanish; let $U \neq 0$. The normalized weights and the filling parameter are

$$p_0 \coloneqq \frac{e^{-\beta\mu}}{2\cosh(\beta\mu) + 2e^{\beta U/2}}, \quad p_1 \coloneqq \frac{e^{\beta U/2}}{2\cosh(\beta\mu) + 2e^{\beta U/2}}, \quad p_2 \coloneqq \frac{e^{\beta\mu}}{2\cosh(\beta\mu) + 2e^{\beta U/2}}, \tag{156}$$

$$\rho \coloneqq \frac{\sinh(\beta\mu)}{\cosh(\beta\mu) + e^{\beta U/2}} = p_2 - p_0. \tag{157}$$

The mirror partner of ν is $-\nu - \omega$, and $\delta_{\nu\nu'}$, $\delta_{\nu(-\nu'-\omega)}$ are Kronecker deltas. Away from half filling the frequencies enter shifted by μ: in the particle–hole channels through $\tilde{\nu} \coloneqq \nu - i\mu$ and $\tilde{\nu}' \coloneqq \nu' - i\mu$, in the particle–particle channels through

$$t_\nu \coloneqq (\nu - i\mu)(\nu + \omega + i\mu), \tag{158}$$

which is $\nu(\nu+\omega)$ at $\mu = 0$. The particle–hole channels $r = \mathrm{d}, \mathrm{m}$ carry the sign $\varepsilon_\mathrm{d} = +1$ (density) and $\varepsilon_\mathrm{m} = -1$ (magnetic). As in Ref. [1], the temperature enters through $\tan(\beta(W+\omega)/4)$ for a square root W (with ω shifted by $2i\mu$ in the particle–hole channels); since $\beta\omega/4 = \pi k/2$, this is $\tan(\beta W/4)$ for even k and $-\cot(\beta W/4)$ for odd k.

F.2 Triplet

With

$$C \coloneqq \frac{U}{2}\rho(2\mu - i\omega) + \frac{U^2}{4}\rho^2, \qquad A_\mathrm{t}^2 \coloneqq -\frac{U^2}{4} - \frac{U}{2}\rho(2\mu - i\omega), \tag{159}$$

for $\nu \neq -\nu - \omega$,

$$\Gamma_\mathrm{t}^{\nu\nu'\omega} = -\frac{\beta(A_\mathrm{t}^2 + C)\left[(\nu - i\mu)^2 + U^2/4\right]\left[(\nu + \omega + i\mu)^2 + U^2/4\right]}{(t_\nu + C)(t_\nu - A_\mathrm{t}^2)}\left[\delta_{\nu\nu'} - \delta_{\nu(-\nu'-\omega)}\right], \tag{160}$$

where $-(A_\mathrm{t}^2 + C) = (U^2/4)(1-\rho^2)$. The factor $t_\nu - A_\mathrm{t}^2$ must be nonzero for every $\nu \neq -\nu - \omega$. At a mirror fixed point, $\nu = -\omega/2$ (odd k), the entire row and column vanish: $\Gamma_\mathrm{t}^{\nu\nu'\omega} = \Gamma_\mathrm{t}^{\nu'\nu\omega} = 0$ for every ν'. At $\mu = 0$, $C = 0$ and $A_\mathrm{t}^2 = -U^2/4$, and this is the triplet case of Eq. (19) of Ref. [1].

F.3 Singlet

With C as for the triplet,

$$B_{\rm s}^2 \coloneqq \frac{U^2}{4}(8p_1-1) - \frac{U}{2}\rho(2\mu - i\omega), \qquad W_{\rm s} \coloneqq \sqrt{4B_{\rm s}^2 + (\omega+2i\mu)^2}, \tag{161}$$

the vertex is

$$\begin{aligned}\Gamma_{\rm s}^{\nu\nu'\omega} = {}& \frac{\beta(B_{\rm s}^2+C)\big[(\nu-i\mu)^2+U^2/4\big]\big[(\nu+\omega+i\mu)^2+U^2/4\big]}{(t_\nu+C)(t_\nu-B_{\rm s}^2)}\Big[\delta_{\nu\nu'}+\delta_{\nu(-\nu'-\omega)}\Big] + 2U \\ & - \frac{2U\Big[\big(B_{\rm s}^2+U^2/4\big)^2 - (U^2/4)(2\mu-i\omega)^2\Big]}{[1+U\tan(\beta(W_{\rm s}+\omega)/4)/W_{\rm s}](t_\nu-B_{\rm s}^2)(t_{\nu'}-B_{\rm s}^2)}.\end{aligned} \tag{162}$$

The factors $t_\nu - B_{\rm s}^2$ must be nonzero for every ν, and so must the thermal denominator. Either sign of $W_{\rm s}$ gives the same value; at $W_{\rm s}=0$ (even k), replace $\tan(\beta(W_{\rm s}+\omega)/4)/W_{\rm s}$ by $\beta/4$. At $\mu=0$, $B_{\rm s}$ is the $B_{\rm s}$ of Ref. [1], and this is the singlet case of its Eq. (19), where the term with $A_{\rm s}=0$ drops out. The lattice-zero continuation is item 1 of Proposition 6.

F.4 Density and magnetic away from half filling

Take $\mu \neq 0$. With

$$q \coloneqq \frac{U^2}{4}(1+4\varepsilon_r p_1), \qquad B_{r,1}^2, B_{r,2}^2 \coloneqq q - \frac{U^2\rho^2}{2} \mp U\sqrt{\frac{U^2}{4}\rho^4 - \rho^2\left(q+\frac{\omega^2}{4}\right) + U^2 p_0 p_2}, \tag{163}$$

$$W_j \coloneqq \sqrt{4B_{r,j}^2+\omega^2}, \qquad \tilde{t}_\nu \coloneqq \tilde{\nu}(\tilde{\nu}+\omega), \tag{164}$$

and for $B_{r,1}^2 \neq B_{r,2}^2$ and $\tilde{t}_\nu \neq B_{r,j}^2$ for every ν and j,

$$\begin{aligned}\Gamma_r^{\nu\nu'\omega} = {}& -\beta\left(\tilde{\nu}^2+\frac{U^2}{4}\right)\left((\tilde{\nu}+\omega)^2+\frac{U^2}{4}\right) \\ & \times\left[\frac{1}{\tilde{t}_\nu - iU\rho(\tilde{\nu}+\omega/2) - U^2\rho^2/4} + \frac{q-\tilde{t}_\nu - iU\rho(\tilde{\nu}+\omega/2)}{\big(\tilde{t}_\nu - B_{r,1}^2\big)\big(\tilde{t}_\nu - B_{r,2}^2\big)}\right]\delta_{\nu\nu'} + \varepsilon_r U \\ & - \frac{i\varepsilon_r U^2\rho}{\tilde{t}_{\nu'}-\tilde{t}_\nu}\left[\frac{(\tilde{\nu}+\omega/2)(\tilde{\nu}^2+U^2/4)((\tilde{\nu}+\omega)^2+U^2/4)}{\big(\tilde{t}_\nu - B_{r,1}^2\big)\big(\tilde{t}_\nu - B_{r,2}^2\big)} - \frac{(\tilde{\nu}'+\omega/2)(\tilde{\nu}'^2+U^2/4)\big((\tilde{\nu}'+\omega)^2+U^2/4\big)}{\big(\tilde{t}_{\nu'} - B_{r,1}^2\big)\big(\tilde{t}_{\nu'} - B_{r,2}^2\big)}\right] \\ & + \frac{i\varepsilon_r U^2\rho}{2}\sum_{j=1}^{2}\left[\frac{W_j\Big[\big(B_{r,j}^2+U^2/4\big)^2 + U^2\omega^2/4\Big]}{\big(B_{r,3-j}^2 - B_{r,j}^2\big)\big(\tilde{t}_\nu - B_{r,j}^2\big)\big(\tilde{t}_{\nu'} - B_{r,j}^2\big)}\right. \\ & \left. \times \frac{q - B_{r,j}^2 - (i/2)U\rho W_j + (\varepsilon_r U^2/2)\big[i\rho\tan\big(\beta\big(W_j+\omega+2i\mu\big)/4\big) + p_0 + p_2\big]}{q - B_{r,j}^2 - (i/2)U\rho W_j - (\varepsilon_r U^2/2)\big[i\rho\tan\big(\beta\big(W_j+\omega+2i\mu\big)/4\big) + p_0 + p_2\big]}\right].\end{aligned} \tag{165}$$

The last denominator must be nonzero for each retained root. For $r = \mathrm{m}$ and $k=0$, omit the root $B_{r,j}^2 = -U^2/4$ from the sum and from this requirement. Either sign of either W_j gives the same vertex. As $\mu \to 0$, $B_{r,1}^2$ and $B_{r,2}^2$ tend to A_r^2 and B_r^2 below, in an order that depends on the signs of U and ε_r.

Diagonal entries. For $\nu = \nu'$, replace the second line (the dense term) by

$$i\varepsilon_r U^2 \rho \left[\frac{N}{2yE} + y \frac{2(\tilde{t}_\nu + U^2/4)E - N(2\tilde{t}_\nu - B_{r,1}^2 - B_{r,2}^2)}{E^2} \right], \tag{166}$$

where $y := \tilde{\nu} + \omega/2$, $E := (\tilde{t}_\nu - B_{r,1}^2)(\tilde{t}_\nu - B_{r,2}^2)$ and $N := (\tilde{t}_\nu + U^2/4)^2 + U^2\omega^2/4 = (\tilde{\nu}^2 + U^2/4)((\tilde{\nu}+\omega)^2 + U^2/4)$. The confluent-root and lattice-zero continuations are items 2 and 3 of Proposition 6.

F.5 Density and magnetic at half filling

Set $\mu = 0$, so that $\rho = 0$ and $\tilde{\nu} = \nu$. With

$$A_r^2 := \frac{U^2}{4}(1 + 2\varepsilon_r), \qquad B_r^2 := \frac{U^2}{4} \frac{3e^{\varepsilon_r \beta U/2} - 1}{e^{\varepsilon_r \beta U/2} + 1}, \qquad W := \sqrt{4B_r^2 + \omega^2}, \tag{167}$$

and for $\nu(\nu+\omega) \neq A_r^2, B_r^2$ for every ν,

$$\begin{aligned} \Gamma_r^{\nu\nu'\omega} = {} & \frac{\beta A_r^2 (\nu^2 + U^2/4)((\nu+\omega)^2 + U^2/4)}{2[\nu(\nu+\omega) - A_r^2]\nu(\nu+\omega)} \left[\delta_{\nu\nu'} - \delta_{\nu(-\nu'-\omega)}\right] \\ & + \frac{\beta B_r^2 (\nu^2 + U^2/4)((\nu+\omega)^2 + U^2/4)}{2[\nu(\nu+\omega) - B_r^2]\nu(\nu+\omega)} \left[\delta_{\nu\nu'} + \delta_{\nu(-\nu'-\omega)}\right] \\ & + \varepsilon_r U - \frac{\varepsilon_r U \left[(B_r^2 + U^2/4)^2 + U^2\omega^2/4\right]}{[1 + \varepsilon_r U \tan(\beta(W+\omega)/4)/W][\nu(\nu+\omega) - B_r^2][\nu'(\nu'+\omega) - B_r^2]}. \end{aligned} \tag{168}$$

This is Eq. (19) of Ref. [1] for $r = \mathrm{d}, \mathrm{m}$, whose prefactors there are $\mathcal{A}_0^r = \mathcal{B}_0^r = 1$, $|\mathcal{B}_2^r| = 1$ and $(\mathcal{B}_1^r)^2 = -\varepsilon_r$. A_r and B_r are those of its Table I, where B_m carries a minus sign that does not matter; its thermal denominator $U\tan(\cdots)/W \pm 1$ is ours up to the sign ε_r. The thermal denominator must be nonzero. Either sign of W gives the same value; at $W = 0$ (even k), replace $\tan(\beta(W+\omega)/4)/W$ by $\beta/4$. The lattice-zero and mirror-fixed-point continuations are item 4 of Proposition 6.

F.6 Doping and Zeeman field

Here h denotes the physical field, **not** the thermal constant h of Eq. 5; the latter is written $\kappa := \beta\sqrt{p_0 p_2}$ throughout the field parts of Appendices F–H. The Hamiltonian is

$$\mathcal{H} = U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) - \mu(n_\uparrow + n_\downarrow - 1) - h(n_\uparrow - n_\downarrow). \tag{169}$$

These formulas follow by the same inversion; the transverse sector is the magnetic-channel problem with $i\omega$ replaced by the complex transfer $i\omega - 2h$ and with the thermal data below. No additional derivation is given here.

F.6.1 Field input and sector conventions

Take real U, μ, h, $\beta > 0$, $a = U/2$, $w = i\omega$, and the frequencies of the input panel. All vertices vanish at $U = 0$. Replace the zero-field weights by

$$Z_f := 2\cosh(\beta\mu) + 2e^{\beta U/2}\cosh(\beta h), \tag{170}$$

$$(p_0, p_\uparrow, p_\downarrow, p_2) := \frac{\left(e^{-\beta\mu}, e^{\beta(U/2+h)}, e^{\beta(U/2-h)}, e^{\beta\mu}\right)}{Z_f}, \tag{171}$$

$$\rho := p_2 - p_0, \quad m_f := p_\uparrow - p_\downarrow, \quad \mu_\sigma := \mu + \sigma h, \quad r_\sigma := \rho - \sigma m_f, \quad \sigma, t \in \{+1, -1\} \equiv \{\uparrow, \downarrow\}. \tag{172}$$

Thus $r_\sigma = 2\langle n_{-\sigma}\rangle - 1$, and

$$G_\sigma(\nu) = \frac{i\nu + \mu_\sigma + a r_\sigma}{(i\nu + \mu_\sigma)^2 - a^2}. \tag{173}$$

The longitudinal sector is the **coupled** connected matrix $\chi_{\sigma t}$ of Eq. 12 in spin and frequency. The transverse $\chi_{\mathrm{T}\,\sigma}$ is the full transform with the same Fourier factor of $\langle T c^\dagger_\sigma(\tau_1) c_{-\sigma}(\tau_2) c^\dagger_{-\sigma}(\tau_3) c_\sigma(0)\rangle$. The oriented opposite-spin pair is $\chi^{\nu\nu'\omega}_{\mathrm{A}\,\sigma} = -\chi^{\nu\nu'(-\omega-\nu-\nu')}_{\mathrm{T}\,\sigma}$; it is not a separate singlet/triplet scalar problem at $h \neq 0$. The equal-spin pair $\chi_{\mathrm{E}\,\sigma}$ is $-1/2$ times the **full**, not connected, same-spin particle–hole four-point function at transfer $-\omega-\nu-\nu'$. The factor $1/2$ removes identical-leg double counting.

All field sectors use $\Gamma_X = \beta^2(\chi_X^{-1} - b_X^{-1})$ ($\eta_X = +1$). Their bubbles are

$$\begin{aligned}
b^{\nu\nu'\omega}_{\mathrm{L},\sigma t} &= -\beta G_\sigma(\nu) G_\sigma(\nu+\omega)\delta_{\sigma t}\delta_{\nu\nu'},\\
b^{\nu\nu'\omega}_{\mathrm{T}\,\sigma} &= -\beta G_\sigma(\nu) G_{-\sigma}(\nu+\omega)\delta_{\nu\nu'},\\
b^{\nu\nu'\omega}_{\mathrm{A}\,\sigma} &= \beta G_\sigma(\nu) G_{-\sigma}(-\nu-\omega)\delta_{\nu\nu'},\\
b^{\nu\nu'\omega}_{\mathrm{E}\,\sigma} &= \frac{\beta}{2} G_\sigma(\nu) G_\sigma(-\nu-\omega)\big[\delta_{\nu\nu'} - \delta_{\nu(-\nu'-\omega)}\big].
\end{aligned} \tag{174}$$

Only the longitudinal inverse couples both spins. For E σ both inverses are restricted to the mirror-odd subspace $\mathcal{M}_-$, then extended by zero; there is no full-space inverse.

To keep the formulas legible, write the physical frequencies through

$$z_c(\nu) \coloneqq c + i(\nu + \omega/2) = i(\nu - ic + \omega/2), \quad S_c(x) \coloneqq -\frac{\beta}{2}\begin{cases}\tanh(\beta(x-c)/2)k & \text{even}\\ \coth(\beta(x-c)/2)\,k & \text{odd}\end{cases}. \tag{175}$$

These are explicit abbreviations, not further unknowns. For comparison with the constants of Ref. [1], a centered root s_j below corresponds to $B_j^2 = -s_j - \omega^2/4$, so $z_c(\nu)^2 - s_j = B_j^2 - (\nu - ic)(\nu - ic + \omega)$.

F.6.2 Transverse spin and opposite-spin pairs

For T $\uparrow$ put $z = z_\mu(\nu)$, $v = z_\mu(\nu')$, and define

$$W \coloneqq i\omega - 2h, \quad q_1 \coloneqq a^2 - W^2/4 - 2a^2(p_\uparrow + p_\downarrow) - aWm_f, \tag{176}$$

$$H_T(x) \coloneqq x^2 + 2a\rho x + q_1, \quad P_T(x) \coloneqq \big[(x+a)^2 - W^2/4\big]\big[(x-a)^2 - W^2/4\big], \tag{177}$$

$$e_T(x) \coloneqq \big(x^2 + q_1\big)^2 - 4a^2\rho^2 x^2 - 16a^4 p_0 p_2 = (x^2 - s_1)(x^2 - s_2), \tag{178}$$

$$s_{1,2} \coloneqq 2a^2\rho^2 - q_1 \pm \sqrt{4a^4\rho^4 - 4a^2\rho^2 q_1 + 16a^4 p_0 p_2}, \quad u_j^2 = s_j. \tag{179}$$

For $\mu \neq 0$, distinct roots off the lattice, and nonzero thermal denominators in the following sum,

$$\begin{aligned}
\Gamma^{\nu\nu'\omega}_{\mathrm{T}\uparrow} = \beta P_T(z)&\left[\frac{1}{(z+a\rho)^2 - \big(W/2 + am_f\big)^2} - \frac{z^2 - 2a\rho z + q_1}{e_T(z)}\right]\delta_{\nu\nu'} - U\\
&+4a^2\rho\frac{zP_T(z)/e_T(z) - vP_T(v)/e_T(v)}{z^2 - v^2}\\
&-4a^2\rho\sum_{j=1}^{2}\frac{u_j P_T(u_j)}{(s_j - s_{3-j})(z^2 - s_j)(v^2 - s_j)}\\
&\times\frac{\beta(s_j - 2a\rho u_j + q_1) + 4a^2\big[\rho S_\mu(u_j) - \beta(p_0+p_2)/2\big]}{\beta(s_j - 2a\rho u_j + q_1) - 4a^2\big[\rho S_\mu(u_j) - \beta(p_0+p_2)/2\big]}.
\end{aligned} \tag{180}$$

On $\nu=\nu'$ the quotient in the second line is $[F_T(z)+zF_{T'}(z)]/(2z)$, where $F_T=\frac{P_T}{e_T}$ and the prime is differentiation in z. Either sign of each u_j gives the same result. At $h=0$, this is Eq. 165 for $r=\mathrm{m}$; at $h=0$, $k=0$ omit the atomic root $s_j=a^2$, whose coefficient vanishes.

At $\mu=0$ use the mirror limit instead. With $\rho=0$, $p_0=p_2$, $\hat{s}=-4a^2p_0-q_1$, $\hat{u}^2=\hat{s}$, and $\hat{D}=\beta+2aS_0(\hat{u})/\hat{u}$,

$$\begin{aligned}\Gamma^{\nu\nu'\omega}_{\mathrm{T}\uparrow}=\beta P_T(z)\left[\frac{1}{z^2-\left(W/2+am_f\right)^2}-\frac{z^2+q_1}{e_T(z)}\right]\delta_{\nu\nu'}-U\\ +4a^2\beta p_0\frac{P_T(z)}{e_T(z)}\delta_{\nu(-\nu'-\omega)}\\ +\frac{2a\beta P_T(\hat{u})}{\hat{D}(z^2-\hat{s})(v^2-\hat{s})}.\end{aligned}\tag{181}$$

Here $e_T(z_n)\neq 0$ and $\hat{D}\neq 0$ are required for the regular formula. At $\hat{u}=0$ and even k, $S_0(\hat{u})/\hat{u}=-\beta^2/4$ by continuity.

The other orientation and the opposite-spin pair are

$$\Gamma_{\mathrm{T}\downarrow}(U,\mu,h)=\Gamma_{\mathrm{T}\uparrow}(U,\mu,-h),\quad \Gamma_{\mathrm{A}\uparrow}(U,\mu,h)=\Gamma_{\mathrm{T}\uparrow}(-U,h,\mu).\tag{182}$$

All frequencies and β are unchanged. Explicitly the second substitution sends $a\to -a$, $\mu\leftrightarrow h$, $\left(p_0,p_\uparrow,p_\downarrow,p_2\right)\to\left(p_\downarrow,p_2,p_0,p_\uparrow\right)$, $\rho\leftrightarrow m_f$, $W\to i\omega-2\mu$, and the bare vertex to $+U$. Thus Eq. 180, or Eq. 181 on $h=0$, writes out the pair vertex too. The down orientation is $\Gamma_{\mathrm{A}\downarrow}(U,\mu,h)=\Gamma_{\mathrm{A}\uparrow}(U,\mu,-h)=\mathcal{M}\Gamma_{\mathrm{A}\uparrow}(U,\mu,h)\mathcal{M}$. Apply spin interchange **before** the particle–hole map: $\Gamma_{\mathrm{A}\downarrow}=\Gamma_{\mathrm{T}\uparrow}(-U,-h,\mu)$, not $\Gamma_{\mathrm{T}\downarrow}(-U,h,\mu)$ at unchanged frequencies.

F.6.3 Equal-spin pairs

Set $b_\sigma=2\mu_\sigma-i\omega$, $x=\mu_\sigma+i\nu$, $t_{\sigma\nu}=x(b_\sigma-x)$, and define

$$\mathcal{P}_\sigma(t)\coloneqq\left(t+a^2\right)^2-a^2b_\sigma^2,\quad \mathcal{G}_\sigma(t)\coloneqq t+ar_\sigma b_\sigma+a^2r_\sigma^2,\quad \mathcal{L}_\sigma(t)\coloneqq t+ar_\sigma b_\sigma+a^2.\tag{183}$$

For $\nu\neq-\nu-\omega$ and $\mathcal{L}_\sigma(t_{\sigma\nu})\neq 0$,

$$\Gamma^{\nu\nu'\omega}_{\mathrm{E}\,\sigma}=-\frac{\beta a^2(1-r_\sigma^2)\left[\left(t_{\sigma\nu}+a^2\right)^2-a^2b_\sigma^2\right]}{2[t_{\sigma\nu}+ar_\sigma b_\sigma+a^2r_\sigma^2][t_{\sigma\nu}+ar_\sigma b_\sigma+a^2]}\left[\delta_{\nu\nu'}-\delta_{\nu(-\nu'-\omega)}\right].\tag{184}$$

At a mirror fixed point the entire row and column vanish. This is $-1/2$ times Eq. 160 with $\mu\to\mu_\sigma$, $\rho\to r_\sigma$.

F.6.4 Coupled longitudinal vertex

Here $z=z_{\mu_\sigma}(\nu)$ and $v=z_{\mu_t}(\nu')$ belong to their own spin lattices. Define

$$\lambda_L\coloneqq U^2,\quad q_0\coloneqq a^2+\omega^2/4,\quad \pi_P\coloneqq p_0p_2,\quad \pi_S\coloneqq p_\uparrow p_\downarrow,\quad d_\pi\coloneqq\pi_P-\pi_S,\tag{185}$$

$$D_t\coloneqq p_2p_t-p_0p_{-t},\quad c_t\coloneqq p_2p_t+p_0p_{-t},\tag{186}$$

$$H_t(x)\coloneqq x^2+2ar_tx+q_0,\quad P_L(x)\coloneqq\left(x^2+q_0\right)^2-4a^2x^2,\tag{187}$$

$$Q_t(x)\coloneqq H_t(-x)H_{-t}(x)H_{-t}(-x)-\lambda_L^2[\pi_PH_{-t}(-x)+\pi_SH_{-t}(x)],\tag{188}$$

$$
\begin{aligned}
E_8(x) \coloneqq\; & H_\uparrow(x)H_\uparrow(-x)H_\downarrow(x)H_\downarrow(-x) \\
& -\lambda_L^2\big\{\pi_S\big[H_\uparrow(x)H_\downarrow(x)+H_\uparrow(-x)H_\downarrow(-x)\big] \\
& \quad +\pi_P\big[H_\uparrow(x)H_\downarrow(-x)+H_\uparrow(-x)H_\downarrow(x)\big]\big\}+\lambda_L^4 d_\pi^2 \\
=\; & \prod_{j=1}^{4}(x^2-s_j).
\end{aligned}
\tag{189}
$$

The s_j now mean the four longitudinal roots, not the two transverse ones. Put $u_j^2 = s_j$ and $E'_{8,j} \coloneqq \prod_{i\neq j}(s_j - s_i)$ (derivative in $s = x^2$, not in x). All mode coefficients are explicit:

$$
\alpha_t(x) \coloneqq D_t S_{\mu_t}(x) - \beta c_t/2, \tag{190}
$$

$$
\alpha_t^P(x) \coloneqq \rho S_{\mu_{-t}}(x) - \beta(p_0+p_2)/2, \quad \alpha_t^S(x) \coloneqq -tm_f S_{\mu_{-t}}(x) - \beta\big(p_\uparrow + p_\downarrow\big)/2, \tag{191}
$$

$$
\begin{aligned}
\big(f_{j\uparrow+}, f_{j\uparrow-}, f_{j\downarrow+}, f_{j\downarrow-}\big) \coloneqq \big(&-\beta Q_\uparrow(u_j), 2\lambda_L^2(s_j+q_0)\alpha_\uparrow(u_j), \\
&-\lambda_L\big[H_\uparrow(-u_j)H_\downarrow(-u_j)+\lambda_L^2 d_\pi\big]\alpha_\downarrow^S(u_j), \\
&-\lambda_L\big[H_\uparrow(-u_j)H_\downarrow(u_j)-\lambda_L^2 d_\pi\big]\alpha_\downarrow^P(u_j)\big),
\end{aligned}
\tag{192}
$$

$$
K_{jt} \coloneqq f_{jt+} + f_{jt-}, \quad \mathfrak{c}_{jt} \coloneqq \frac{2\lambda_L^2 D_t P_L\big(u_j\big)\big(s_j+q_0\big)}{\beta^2 E'_{8,j} K_{jt}}. \tag{193}
$$

If this quotient is $0/0$ but $K_{j,-t} \neq 0$, use the equal, regular quotient

$$
\mathfrak{c}_{jt} = \frac{-\lambda_L P_L\big(u_j\big)\big[\rho\Xi_j + tm_f\Pi_j\big]}{\beta^2 E'_{8,j} K_{j,-t}}, \tag{194}
$$

$$
\Xi_j \coloneqq \big(s_j+q_0\big)^2 - 4a^2 r_\uparrow r_\downarrow s_j - \lambda_L^2 d_\pi, \quad \Pi_j \coloneqq \big(s_j+q_0\big)^2 + 4a^2 r_\uparrow r_\downarrow s_j + \lambda_L^2 d_\pi. \tag{195}
$$

For $U \neq 0$, $\mu h(\mu^2 - h^2) \neq 0$, simple nonzero roots off both spin lattices, $P_L\big(u_j\big) \neq 0$ at non-atomic roots, and $\big(K_{j\uparrow}, K_{j\downarrow}\big) \neq (0,0)$, the vertex is

$$
\begin{aligned}
\Gamma^{\nu\nu'\omega}_{\mathrm{L},\sigma t} = \;& \beta P_L(z)\left[\frac{1}{\big(z+ar_t\big)^2+\omega^2/4} - \frac{Q_t(z)}{E_8(z)}\right]\delta_{\sigma t}\delta_{\nu\nu'} + U\delta_{\sigma(-t)} \\
& +\delta_{\sigma t}\frac{2\lambda_L^2 D_t(v^2+q_0)P_L(v)}{E_8(v)(z+v)} \\
& -\delta_{\sigma(-t)}\frac{\lambda_L P_L(v)}{E_8(v)}\left[\frac{tm_f[H_t(-v)H_{-t}(-v)+\lambda_L^2 d_\pi]}{z-v}\right. \\
& \quad \left.+\frac{\rho[H_t(-v)H_{-t}(v)-\lambda_L^2 d_\pi]}{z+v}\right] \\
& -\beta^2\sum_j \mathfrak{c}_{jt}\frac{1}{v^2-s_j}\left[\frac{f_{j\sigma+}}{z-u_j}+\frac{f_{j\sigma-}}{z+u_j}\right].
\end{aligned}
\tag{196}
$$

At $k=0$ omit the atomic root $s_j = a^2$ from the sum and the nonvanishing requirements: its coefficient is zero. The inverse is two-sided, $\Gamma^{\nu\nu'\omega}_{\mathrm{L},\sigma t} = \Gamma^{\nu'\nu\omega}_{\mathrm{L},t\sigma}$, and neither square-root signs nor a common rescaling of each mode affect the answer.

F.6.5 Reductions, continuous extensions and exceptional sets

At $h = 0$, all functions on the right below are evaluated at (U, β, μ):

$$\Gamma_{\mathrm{L},\sigma\sigma} = (\Gamma_{\mathrm{d}} + \Gamma_{\mathrm{m}})/2, \quad \Gamma_{\mathrm{L},\sigma(-\sigma)} = (\Gamma_{\mathrm{d}} - \Gamma_{\mathrm{m}})/2, \tag{197}$$

$$\Gamma_{\mathrm{T}\,\sigma} = \Gamma_{\mathrm{m}}, \quad \Gamma_{\mathrm{A}\,\sigma} = (\Gamma_{\mathrm{s}} - \Gamma_{\mathrm{t}})/2, \quad \Gamma_{\mathrm{E}\,\sigma} = -\Gamma_{\mathrm{t}}/2. \tag{198}$$

The particle–hole (Shiba) map $c_\downarrow \to c_\downarrow^\dagger$, $(U, \mu, h) \to (-U, h, \mu)$ [2] gives the $\mu = 0$ values. Write D for the zero-field source $(-U, \beta, h)$, $\varepsilon_\uparrow = 1$, $\varepsilon_\downarrow = -1$, $R_\uparrow \nu = \nu$, $R_\downarrow \nu = -\nu - \omega$. Then

$$\Gamma^{\nu\nu'\omega}_{\mathrm{L},\sigma t} = \frac{\varepsilon_\sigma \varepsilon_t}{2}\left[\Gamma^D_{\mathrm{d}}(R_\sigma\nu, R_t\nu', \omega) + \varepsilon_\sigma\varepsilon_t\Gamma^D_{\mathrm{m}}(R_\sigma\nu, R_t\nu', \omega)\right], \tag{199}$$

$$\Gamma_{\mathrm{T}\uparrow} = (\Gamma^D_{\mathrm{s}} - \Gamma^D_{\mathrm{t}})/2, \quad \Gamma_{\mathrm{A}\uparrow} = \Gamma^D_{\mathrm{m}}, \quad \Gamma_{\mathrm{E}\uparrow} = -\Gamma^D_{\mathrm{t}}/2. \tag{200}$$

For down orientation use the source $(-U, \beta, -h)$. In the charge/S_z basis the longitudinal map is the orthogonal similarity $\mathcal{S}\,\mathrm{diag}(\Gamma^D_{\mathrm{d}}, \Gamma^D_{\mathrm{m}})\mathcal{S}$ with $\mathcal{S} = \left(\begin{smallmatrix} \mathcal{M}_- & \mathcal{M}_+ \\ \mathcal{M}_+ & \mathcal{M}_- \end{smallmatrix}\right)$; separate inversion of its diagonal blocks is incorrect.

Longitudinal axes. In Eq. 196 replace the three kernels by their limits where their denominators vanish:

$$\frac{D_t}{z + v} \to \beta p_0 p_{-t}\delta_{\nu(-\nu'-\omega)} \quad (\mu_t = 0), \tag{201}$$

$$\frac{\rho}{z + v} \to \beta p_0 \delta_{\nu(-\nu'-\omega)} \quad (\mu = 0), \quad \frac{m_f}{z - v} \to \sigma\beta p_\downarrow \delta_{\nu\nu'} \quad (h = 0, \sigma = -t). \tag{202}$$

Together with Eq. 195 these give the continuous extension on $\mu = \pm h \neq 0$. Agreement there with the vertex is **numerically verified, not derived**. On $h = 0$ and $\mu = 0$ the independently known values are Eq. 198 and Eq. 200. At $\mu = h = 0$ use those values, rather than the simultaneous 0/0 mode coefficients.

Other exceptional sets. For the transverse formula $W = \pm 2a$ merely makes P_T have a double zero; $P_T(u_j) = 0$ alone is harmless. Confluent roots and lattice roots require limiting formulas, not direct substitution. For T their removable continuations are those of Proposition 6 with the present data (at $\mu = 0$, the mirror-even lattice cancellation and mirror-odd genuine pole must be distinguished). For L, confluence can be treated by confluent divided differences of the root sum away from thermal poles; this continuation is numerically checked, not established here with uniform bounds. Axis lattice roots can be genuine divergences: at $\beta = 2$, $\mu = h = 0.3$, $k = 0$ an example occurs near $U = 2.3120511518839$ (numerical). Off-axis lattice cancellations are expected from coincident lattice and thermal values, but are not established here. Non-atomic zeros of $P_L(u_j)$, $u_j = 0$, and vanishing adjugate modes are excluded from the stated longitudinal domain; a vanishing mode requires another adjugate column. No general claim about those isolated extensions is made. Thermal poles and zeros of $\mathcal{L}_\sigma$ are the genuine divergences discussed in Appendix G.5; averaging across them is not a regularization.

G Vertex divergences

The vertex $\Gamma_r = \beta^2(\chi_r^{-1} - \eta_r \boldsymbol{B}^{-1})$ diverges where χ_r has a zero mode; the entries of $\boldsymbol{B}^{-1}$ are finite, since G has no zeros on the lattice. These divergences were found analytically at half filling [1], located as zeros of $\det \chi_r$ on finite boxes [3], and mapped away from half filling through the eigenvalues of χ_r on finite boxes [2,4]. Section 4.4 lists the genuine poles of the closed forms. Here we give a null vector of χ_r at each of them, derive the conditions at $\omega = 0$, and plot them (Figure 4). The conditions are derived under the nondegeneracy assumptions of Section 4.4. The exceptional points, the map, and the statements about it are computed from these conditions and checked against finite boxes (Appendix D); they are numerical.

G.1 Null vectors

At each genuine pole, $\chi_r \boldsymbol{g} = 0$ for the following vector $\boldsymbol{g}$ (all sums converge absolutely):

- **Density and magnetic**, $\mu \neq 0$, $K_j = 0$: $g_l = 1/(z_l^2 - s_j)$. At $K_j = 0$, $\lambda\alpha_j = -\beta H(-u_j)$, so the mode Eq. 122 is $\varphi_j = 2\beta u_j H(-u_j)\, \boldsymbol{g}$, and Eq. 125 gives $\chi_r \varphi_j = 0$. At a genuine pole, $u_j H(-u_j) \neq 0$ (Section 4.4).
- **Singlet**, $K_\mathrm{s} = 0$ with $\mathcal{P}(-A_\mathrm{s}) \neq 0$: $g_l = 1/(t_l + A_\mathrm{s})$. In the proof of Theorem 3, the matrix $\Xi = \mathrm{diag}(-1/a, K_\mathrm{s}/(a d_+ d_-))$ becomes singular, and its null direction corresponds to $\boldsymbol{g}$.
- **Triplet**, $t_n + a\rho b + a^2 = 0$ with $n \neq \overline{n}$: $\boldsymbol{g} = \boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}$. By Proposition 11 the mirror-odd vector on the orbit has the eigenvalue $L_\mathrm{t}(t_n) = -\beta(t_n + a\rho b + a^2)/(2\mathcal{P}(t_n)) = 0$.
- **Half filling**, $\beta H(z_n) = -\lambda h$ with $n \neq \overline{n}$: $\boldsymbol{g} = \boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}$ (Section 4.4). For $\hat{D} = 0$, $g_l = \hat{g}(z_l)$, the limit $\mu \to 0$ of the density and magnetic case (checked numerically, Appendix D).

We do not show that these vectors span the kernel. Following Ref. [1], a divergence is *localized* if its null vector has finite support, and *global* otherwise. Only the triplet divergences and the mirror-odd ones at half filling are localized. At half filling the localized null vectors are mirror-odd and the global ones mirror-even: these are the antisymmetric and symmetric divergences of Ref. [2]. Away from half filling $\boldsymbol{g}$ is neither. A global null vector is concentrated where its denominator, $t_l + A_\mathrm{s}$ or $z_l^2 - s_j$, is small. This is how the localized divergences continue away from half filling (below).

G.2 Static transfer

From now on $\omega = 0$ ($k = 0$); at $\omega \neq 0$, localized divergences occur in more channels even at half filling [1]. The zero-field discussion below is static; Appendix G.5 returns to arbitrary transfer. At $\omega = 0$, $b = 2\mu$ and $t_n = |x_n|^2$ are real, so χ_s and χ_t are real symmetric matrices, while χ_d and χ_m are centrohermitian, $\chi_{r,\overline{nm}} = \chi^*_{r,nm}$ [2]. Their eigenvalues are therefore real or come in complex-conjugate pairs. A conjugate pair can change the sign of its real part without vanishing (a *pseudo-divergence* [2]); Γ_r stays finite there. Up to a factor β^3, χ_r depends on β, U and μ only through βU and $\beta\mu$ [2].

Triplet. Since $t_n = \mu^2 + \nu_n^2$ and $|\rho| < 1$,

$$t_n + a\rho b + a^2 = (\mu + a\rho)^2 + \nu_n^2 + a^2(1 - \rho^2) > 0, \tag{203}$$

so there are no triplet divergences.

Singlet. With $\xi^2 = \mu^2 + 2a\rho\mu + a^2(1 - 8p_1)$, which is real, Eq. 30 gives $\mathcal{T}_0(-A_\mathrm{s}) = \beta \tanh(\beta\xi/2)/(2\xi)$, so

$$K_\mathrm{s} \cosh\frac{\beta\xi}{2} = \cosh\frac{\beta\xi}{2} + a\frac{\sinh(\beta\xi/2)}{\xi}, \tag{204}$$

an even, hence real, function of ξ. Γ_s diverges where it vanishes, except where $\mathcal{P}(-A_\mathrm{s}) = d_+ d_-$ (Eq. 62) vanishes too. For $U \neq 0$ and $b \neq \pm 2a$, $d_\pm = 0$ in fact forces $K_\mathrm{s} = 0$: if $d_+ = 0$, then $-A_\mathrm{s} = t_+$, and Eq. 60 with the value of S_- in Lemma 8 gives

$$K_{\mathrm{s}} = 1 - \frac{4aS_-}{\beta(2a-b)} = \frac{4ap_1 - b(p_1+p_2)}{(2a-b)(p_1+p_2)} = -\frac{d_+}{2a(2a-b)(p_1+p_2)} = 0, \tag{205}$$

and likewise for d_-. For $\mu > 0$ this happens on one curve, at $U < 0$, where $d_- = -2a[4ap_1 + 2\mu(p_0 + p_1)] = 0$; for $\mu < 0$ on its mirror image, where $d_+ = 0$. There the separable term of Eq. 34 is 0/0. Numerically it has a finite limit, and on finite boxes no eigenvalue of χ_{s} with an eigenvector aligned with $\boldsymbol{g}$ becomes small (Appendix D): this curve is not a divergence.

Density and magnetic, $\mu \neq 0$. The roots $s_{1,2}$ of Eq. 36 are real or complex conjugate. Since $K(\overline{u}) = \overline{K(u)}$ for the function $K(u) \coloneqq \beta H(-u) + \lambda\alpha_0(u)$, conjugate roots give $K_1 = 0$ only together with $K_2 = 0$. That is two real conditions on the two parameters, which generically hold at isolated points; we do not pursue them. Let s_j be real and $u \coloneqq u_j$. By Eq. 98,

$$\alpha_0(\pm u) = -h\frac{\cosh(\beta(u \pm \mu)/2)}{\cosh(\beta(u \mp \mu)/2)}, \tag{206}$$

so the parts of α_0 even and odd in u are

$$\alpha_{\mathrm{e}} = -h\frac{1 + \cosh(\beta\mu)\cosh(\beta u)}{\cosh(\beta u) + \cosh(\beta\mu)}, \qquad \alpha_{\mathrm{o}} = -h\frac{\sinh(\beta\mu)\sinh(\beta u)}{\cosh(\beta u) + \cosh(\beta\mu)}. \tag{207}$$

With $h\sinh(\beta\mu) = \beta\rho/2$ (Eq. 7), the even and odd parts of $K_j = K(u)$ are

$$K_{\mathrm{e}} = \beta\big(s_j + q_1\big) + \lambda\alpha_{\mathrm{e}}, \qquad K_{\mathrm{o}} = -\frac{2a\beta\rho\, u\, \Phi\big(s_j\big)}{\cosh(\beta u) + \cosh(\beta\mu)}, \tag{208}$$

where

$$\Phi(s) \coloneqq \cosh(\beta\mu) + \cosh(\beta u) + \varepsilon_r a\frac{\sinh(\beta u)}{u} \qquad (u^2 = s). \tag{209}$$

Φ is even in u, hence a real function of real s; for $s = -y^2$ it reads $\cosh(\beta\mu) + \cos(\beta y) + \varepsilon_r a\sin(\beta y)/y$. At the root, Eq. 36 and Eq. 98 give $\beta^2 H(u)H(-u) = \lambda^2 h^2 = \lambda^2\alpha_0(u)\alpha_0(-u)$, and hence

$$\beta H(-u)K(-u) = \lambda\alpha_0(-u)K(u). \tag{210}$$

So $K_j = 0$ if and only if $K(u) = K(-u) = 0$, that is, if and only if $K_{\mathrm{e}} = K_{\mathrm{o}} = 0$; for $u \neq 0$ the latter means $\Phi\big(s_j\big) = 0$. Conversely, if $\Phi\big(s_j\big) = 0$, then $K_{\mathrm{o}} = 0$ and $K(u) = K(-u) = K_{\mathrm{e}}$, and the last identity becomes

$$K_{\mathrm{e}}A_{\mathrm{e}} = 0, \qquad A_{\mathrm{e}} \coloneqq \beta\big(s_j + q_1\big) - \lambda\alpha_{\mathrm{e}}. \tag{211}$$

Along each analytic arc of the curve $\Phi\big(s_j\big) = 0$ in the $(\beta\mu, \beta U)$ plane, one factor therefore vanishes identically. The arcs with $K_{\mathrm{e}} = 0$ are divergences; on the others $K_j \neq 0$ and the vertex is finite. Both factors vanish only where two such arcs cross, and there $K_j = 0$. With Eq. 207, Γ_{d} and Γ_{m} thus diverge where, for a real root $s_j \neq 0$,

$$\Phi\big(s_j\big) = 0 \quad \text{and} \quad \beta\big(s_j + q_1\big)\big(\cosh\big(\beta u_j\big) + \cosh(\beta\mu)\big) = \lambda h\big(1 + \cosh(\beta\mu)\cosh\big(\beta u_j\big)\big). \tag{212}$$

On $\Phi\big(s_j\big) = 0$ the two sides of the second equation agree up to sign, as $K_{\mathrm{e}}A_{\mathrm{e}} = \beta^2\big(s_j + q_1\big)^2 - \lambda^2\alpha_{\mathrm{e}}^2$. So the second condition only selects the sign: $\big(s_j + q_1\big)\,\varepsilon_r\big(1 + \cosh(\beta\mu)\cosh\big(\beta u_j\big)\big)/\big(\cosh\big(\beta u_j\big) + \cosh(\beta\mu)\big) \geq 0$, with equality exactly at the crossings. By Lemma 8,

$$\sum_n \frac{1}{z_n^2 - s} = \frac{S_0(u) - S_0(-u)}{2u} = -\frac{\beta\sinh(\beta u)}{2u(\cosh(\beta u) + \cosh(\beta\mu))}, \tag{213}$$

an absolutely convergent sum. So $\Phi(s) = (\cosh(\beta u) + \cosh(\beta\mu))\Big[1 - (2\varepsilon_r a/\beta)\sum_n \left(z_n^2 - s\right)^{-1}\Big]$, and the first condition in Eq. 212 reads

$$\beta = 2\varepsilon_r a \sum_n \frac{1}{z_n^2 - s_j}, \tag{214}$$

the counterpart of $\hat{D} = 0$ (Eq. 42) with s_j in place of $\hat{s}$.

Half filling. At $\mu = 0$,

$$\Phi(s) = 2\cosh^2\frac{\beta u}{2}\left[1 + \varepsilon_r a \frac{\tanh(\beta u/2)}{u}\right], \tag{215}$$

and the bracket is $\hat{D}/\beta$ with $\hat{u}$ replaced by u. The zeros of $\cosh(\beta u/2)$ are the lattice points $s = z_n^2$. For $s_j = \hat{s}'$ they are the localized divergences $\beta H(z_n) = -\lambda h$; for $s_j = \hat{s}$ they are the removable cases of item 4 of Proposition 6. The zeros of the bracket at $s_j = \hat{s}$ are the global divergences $\hat{D} = 0$. These are Eqs. (20) and (22) of Ref. [1] at $\omega = 0$. Localized divergences occur only in the density channel, at $\nu_n^2 = 3a^2$, since $\hat{s}' = -a^2(1 + 2\varepsilon_r)$ at $k = 0$.

For $\mu \neq 0$, $\cosh(\beta u) + \cosh(\beta\mu) > 0$ for real s, so no localized divergences remain. The localized lines continue as zeros of Φ. Along them g_j is concentrated on the orbit $\{n, \overline{n}\}$ with $z_n^2 \approx s_j$: as $\mu \to 0$, $z_n^2 - s_j \approx 2i\mu\nu_n$ and $z_{\overline{n}}^2 - s_j \approx -2i\mu\nu_n$, so $\boldsymbol{g}$ tends to $\boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}$ up to normalization.

Exceptional points. For a finite complex symmetric matrix whose kernel is spanned by $\boldsymbol{g}$, the zero eigenvalue is defective, an exceptional point, exactly when $\boldsymbol{g}^T\boldsymbol{g} = 0$. We use the corresponding condition for the null vectors $\boldsymbol{g}_j$ as the criterion for the exceptional points of χ_r on the density and magnetic lines:

$$\boldsymbol{g}_j^T\boldsymbol{g}_j = \sum_n \frac{1}{\left(z_n^2 - s_j\right)^2} = \frac{\mathrm{d}}{\mathrm{d}s}\sum_n \frac{1}{z_n^2 - s}\,\Big|_{s=s_j} = 0. \tag{216}$$

We do not prove this for the infinite matrix. On finite boxes, two eigenvalues of χ_r with eigenvectors aligned with $\boldsymbol{g}_j$ form a near-zero pair of nearly equal size there, which shrinks as the box grows (Appendix D). Generically a pair of real eigenvalues meets at zero and continues as a conjugate pair; the pseudo-divergence lines of Ref. [2] start at these points. The singlet has no exceptional points, as its $\boldsymbol{g}$ is real and $\boldsymbol{g}^T\boldsymbol{g} > 0$.

G.3 The divergence map

Figure 4 shows the lines in the window $|\beta\mu| \leq 4$, $|\beta U| \leq 25$. The statements below describe this window; the numbers are from `doped-gamma/check_divergences.jl` (Appendix D).

- **Density, $U > 0$.** Each localized line and the global line above it are the two halves of one closed loop. On the first loop, the exceptional points are at $|\beta\mu| = 0.564$, $\beta U = 4.33$ (a doping of $\rho = 0.060$). The loop ends at $|\beta\mu| = 0.569$, $\beta U = 4.45$, where $s_1 = s_2$. For $\beta U \to \infty$ at fixed $\beta\mu$, both roots tend to $-3a^2$, and $\Phi(-3a^2) = \cosh(\beta\mu) + \cos\left(\sqrt{3}\beta a\right) + \sin\left(\sqrt{3}\beta a\right)/\sqrt{3}$ vanishes for some a only if $\cosh(\beta\mu) \leq 2/\sqrt{3}$. Likewise, with $q_1 \to 3a^2$ and $\rho^2 = 4p_0p_2\sinh^2(\beta\mu)$, the roots are real only if $\sinh^2(\beta\mu) \leq 1/3$. Both give $|\beta\mu| \leq \operatorname{arsinh}\left(1/\sqrt{3}\right) = 0.549$: to leading order, the loops shrink to this range.
- **Singlet, $U > 0$.** The lines start at the global density lines, with which they coincide at half filling [1], and rise with $|\beta\mu|$.
- **Magnetic, $U > 0$.** There are no divergences in the window, as at half filling [1].
- **$U < 0$.** The density lines start at the localized points, the magnetic ones at the global points of half filling. The first exceptional points are at $|\beta\mu| = 1.121$ (density) and 0.971 (magnetic); beyond

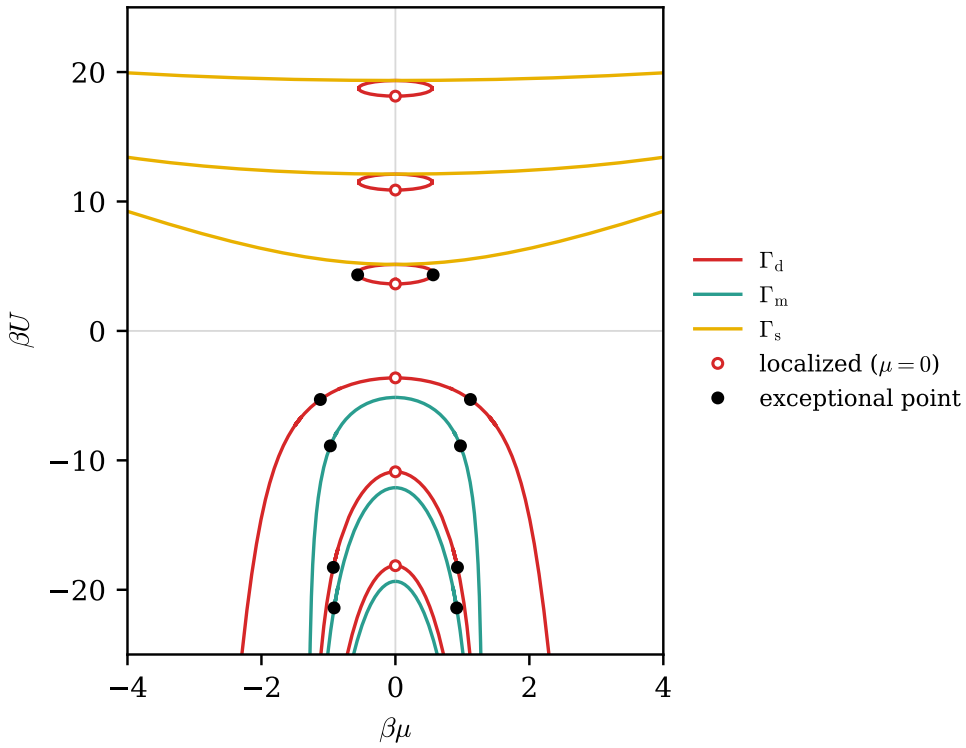


Figure 4: Divergences of $\Gamma_{\rm d}$ (red), $\Gamma_{\rm m}$ (green) and $\Gamma_{\rm s}$ (yellow) at $\omega = 0$, from Eq. 212 and Eq. 204. The map is symmetric under $\mu \to -\mu$. Open circles: the localized divergences $\beta U = \pm 2\pi(2n+1)/\sqrt{3}$ at half filling. Black dots: exceptional points, Eq. 216; on the second and third loop at $U > 0$ they lie near the loop ends, $|\beta\mu| \approx 0.549$, and are not resolved at this scale. Pseudo-divergences [2], where Γ_r is finite, are not shown.

them the lines leave the window at $\beta U = -25$. A line passes from one root to the other where $s_1 = s_2$, for example at $|\beta\mu| = 1.453$, $\beta U = -6.97$ on the first density line.

This is the structure that Ref. [2] found numerically (its Fig. 2a, where $\delta\mu$ is our μ). It reports a doping of about 6% at the first exceptional points for $U > 0$, and Ref. [4] gives them at $\beta\delta\mu \approx 0.6$, 1.12 and 0.975, in agreement with ours.

G.4 In the notation of Ref. [1]

Appendix F uses the notation of Ref. [1]. It relates to that of the main text as follows (the symbols on the right are local to Appendix F and to this section):

Main text	**Ref. [1], Appendix F**
$\Gamma_{r,nm}$, $\delta_{m\overline{n}}$	$\Gamma_r^{\nu\nu'\omega}$ with $\nu = \nu_n$, $\nu' = \nu_m$, $\omega = \omega_k$; $\delta_{\nu(-\nu'-\omega)}$
x_n, $\overline{x}_n$, t_n	$i(\nu - i\mu)$, $-i(\nu + \omega + i\mu)$, t_ν
z_n, $z_n^2 + \omega^2/4$	$i(\tilde{\nu} + \omega/2)$, $-\tilde{\nu}(\tilde{\nu} + \omega)$
s_j, u_j	$-B_{r,j}^2 - \omega^2/4$, $(i/2)W_j$
$\hat{s}'$, $\hat{s}$ $(\mu = 0)$	$-A_r^2 - \omega^2/4$, $-B_r^2 - \omega^2/4$
$-A_{\rm s}$, ξ	$B_{\rm s}^2$, $(i/2)W_{\rm s}$
$\mathcal{G}(t_n)$, $t_n + a\rho b + a^2$	$t_\nu + C$, $t_\nu - A_{\rm t}^2$
$T(iW/2 - \mu)$	$i\tan(\beta(W + \omega + 2i\mu)/4)$

Table 2: Main-text notation and that of Ref. [1] and Appendix F, with $\tilde{\nu} = \nu - i\mu$.

So $z_n^2 - s_j = B_{r,j}^2 - \tilde{\nu}(\tilde{\nu} + \omega)$: the denominators of the separable terms are those of Ref. [1] with $\nu \to \tilde{\nu}$ and with B_r replaced by the two constants $B_{r,j}$. At half filling these are A_r and B_r; away from it they are the two roots of one quadratic, and the localized and global divergences of half filling become the zeros of one function.

At $\omega = 0$, with $s_j = -B_{r,j}^2$, $u_j = iB_{r,j}$ and $q = (U^2/4)(1 + 4\varepsilon_r p_1)$, the results of this appendix read:

- **Null vectors:** $g(\nu) = 1/\big((\nu - i\mu)^2 - B_{r,j}^2\big)$ (density and magnetic), $1/(t_\nu - B_{\mathrm{s}}^2)$ (singlet), and $\boldsymbol{e}_\nu - \boldsymbol{e}_{-\nu-\omega}$ (triplet; localized divergences at half filling).
- **Density and magnetic**, Eq. 212:

$$\begin{aligned} &\cosh(\beta\mu) + \cos\big(\beta B_{r,j}\big) + \varepsilon_r U \frac{\sin\big(\beta B_{r,j}\big)}{2B_{r,j}} = 0 \quad \text{and} \\ &\big(q - B_{r,j}^2\big)\big(\cos\big(\beta B_{r,j}\big) + \cosh(\beta\mu)\big) = \varepsilon_r U^2 \sqrt{p_0 p_2}\big(1 + \cosh(\beta\mu)\cos\big(\beta B_{r,j}\big)\big). \end{aligned} \tag{217}$$

  The first condition is $\beta = \varepsilon_r U \sum_\nu \big(B_{r,j}^2 - (\nu - i\mu)^2\big)^{-1}$. Both sides are even in $B_{r,j}$, hence real for real $B_{r,j}^2$.
- **Exceptional points**, Eq. 216: $\sum_\nu \big((\nu - i\mu)^2 - B_{r,j}^2\big)^{-2} = 0$.
- **Half filling:** the first condition becomes $2\cos^2(\beta B/2)[1 + \varepsilon_r U \tan(\beta B/2)/(2B)] = 0$. Its first factor vanishes for $B^2 = \nu^2$, which for $B = A_r$ is the localized condition $\nu(\nu + \omega) = A_r^2$ of Eq. (20) of Ref. [1]. Its second factor, for $B = B_r$, is the global condition of Eq. (22) there, up to the sign ε_r.
- **Singlet**, Eq. 204, with $B_{\mathrm{s}}^2 = (U^2/4)(8p_1 - 1) - U\rho\mu$:

$$\cos\frac{\beta\sqrt{B_{\mathrm{s}}^2 - \mu^2}}{2} + U\frac{\sin\big(\beta\sqrt{B_{\mathrm{s}}^2 - \mu^2}/2\big)}{2\sqrt{B_{\mathrm{s}}^2 - \mu^2}} = 0, \tag{218}$$

  except on the curve where $d_\mp = 0$.

G.5 Divergences with doping and field

This section restores arbitrary transfer $\omega = 2\pi k/\beta$ and uses the field notation of Appendix F.6, in particular h for the physical field and $\kappa = \beta\sqrt{p_0 p_2}$. Conditions below concern actual square-summable null vectors, not the accumulation of susceptibility eigenvalues at zero as the frequency tends to infinity. They hold on the regular domains stated there, with thermal poles approached from nonsingular parameters. Numerical statements and the unresolved longitudinal extensions are explicitly distinguished.

G.5.1 Transverse spectrum, selection and null vectors

For $u^2 = s$ define the even functions

$$D_k(s) := \cosh(\beta u) + (-1)^k \cosh(\beta\mu), \quad R_k(s) := \frac{\cosh(\beta\mu)\cosh(\beta u) + (-1)^k}{D_k(s)}, \tag{219}$$

$$\Phi_k^T(s) := D_k(s) - a\frac{\sinh(\beta u)}{u}, \quad K_e^T(s) := \beta(s + q_1) + 4a^2\kappa R_k(s), \tag{220}$$

$$\ell_T(s) := -\frac{K_e^T(s)}{\Big[(s + a^2 - W^2/4)^2 - 4a^2 s\Big]}. \tag{221}$$

Use $\sinh(\beta u)/u = \beta$ at $u = 0$. Suppose $\mu U W \neq 0$ and $P_T(u) \neq 0$ at every zero of Φ_k^T (with $D_k \neq 0$). Then

$$\operatorname{spec} \chi_{\mathrm{T}\uparrow} = \{0\} \cup \big\{\ell_T(s) : \Phi_k^T(s) = 0\big\}, \quad g_s(\nu) = \frac{1}{z_\mu(\nu)^2 - s}. \tag{222}$$

The eigenvectors and their coalescence loci depend on μ, U, β and the parity of k, but **not on the field**. Only the eigenvalues move with h through q_1 and P_T. The reason is the same Cauchy-column calculation as in the magnetic problem: $\chi_{\mathrm{T}\uparrow}\boldsymbol{g}_s = \ell_T(s)\boldsymbol{g}_s + \Phi_k^T(s)\boldsymbol{w}_s$ for a regular remainder; the complex transfer changes the polynomials but not the thermal lattice sum. Compactness gives the accumulation point zero; the same pole calculation exhausts the nonzero spectrum under the stated exclusions.

Consequently the transverse divergence condition is

$$\Phi_k^T(s) = 0, \quad K_e^T(s) = 0, \quad P_T(u) \neq 0, \tag{223}$$

and s is then a root of e_T. For a simple zero the null vector is $\boldsymbol{g}_s$ above. Equivalently at a regular root s_j,

$$K_j^T := \beta H_T(-u_j) - 4a^2\left[\rho S_\mu(u_j) - \beta(p_0+p_2)/2\right] = 0. \tag{224}$$

With $\alpha(u) := \rho S_\mu(u) - \beta(p_0+p_2)/2$, $\alpha(u)\alpha(-u) = \kappa^2$ and $e_T(u_j) = 0$ give the same even/odd selection as in Eq. 212. On $\Phi_k^T(s_j) = 0$,

$$K_e^T(s_j)A_e^T(s_j) = 0, \quad A_e^T(s) := \beta(s+q_1) - 4a^2\kappa R_k(s). \tag{225}$$

Only the $K_e^T = 0$ branch is selected; $A_e^T = 0$ alone cancels. For real s_j at $k = 0$ the sign test is $(s_j + q_1)R_k(s_j) \leq 0$. The pole is genuine provided $u_j P_T(u_j) H_T(-u_j) \neq 0$.

Atomic cancellations. At $\Phi_k^T(s) = P_T(u) = 0$, $s = (a \pm W/2)^2$, the equations $K_e^T = 0$ and $e_T(u) = 0$ follow automatically but imply no zero mode: the eigenvalue formula is $0/0$. At $k = 0$ these curves continue the cancelled atomic pair at $h = 0$. They must not be counted as divergences without testing the limiting eigenvalue.

Exceptional points. Multiple zeros of Φ_k^T coalesce eigenvectors. For a simple null direction, the bilinear self-overlap criterion is

$$\sum_n \frac{1}{\left(z_\mu(\nu_n)^2 - s\right)^2} = T_{\mu'}(s) = 0, \quad T_c(s) := -\frac{\beta\sinh(\beta u)}{2u[\cosh(\beta u) + (-1)^k\cosh(\beta c)]}. \tag{226}$$

The prime is differentiation in s. This follows by differentiating the absolutely convergent sum $T_c = \sum_n \left(z_c(\nu_n)^2 - s\right)^{-1}$; as in Eq. 216, its interpretation as a defective zero is checked on finite boxes, not proved here for the infinite operator. At fixed $\beta h = 0.5$ the first static transverse exceptional divergence is near $(\beta\mu, \beta U) = (0.97773, -8.96512)$ (numerical), where the divergence curve crosses a field-independent coalescence locus.

At $k = 0$ or $h = 0$ the transverse spectrum is conjugation-closed. At general k, $\operatorname{spec}\chi_{\mathrm{T}\uparrow}(-\omega) = \overline{\operatorname{spec}\chi_{\mathrm{T}\uparrow}(\omega)}$ and $\operatorname{spec}\chi_{\mathrm{T}\downarrow} = \overline{\operatorname{spec}\chi_{\mathrm{T}\uparrow}}$, so both orientations have the same divergences. For a real zero s,

$$\operatorname{Im} K_e^T(s) = \beta\omega(h - am_f). \tag{227}$$

Thus at $\omega \neq 0$ a real-root divergence requires $h = am_f$ (and $U > 0$ if $h \neq 0$). For complex roots the condition is complex and generically has codimension two. A numerical example is $k = 2$, $\beta h = 1$, $(\beta U, \beta\mu) \approx (-16.69512285, 0.75904959)$, $\beta^2 s \approx -101.558 + 1.632i$; it is an isolated divergence in that fixed-field slice, not a whole line.

G.5.2 Opposite-spin pairing

Apply $(U, \mu, h) \to (-U, h, \mu)$ to the transverse spectrum, null vector, branch selection, exclusions and exceptional-point condition. Explicitly,

$$\Phi_k^A(s) = \cosh(\beta u) + (-1)^k\cosh(\beta h) + a\frac{\sinh(\beta u)}{u}, \tag{228}$$

$$W_A = i\omega - 2\mu, \quad q_1^A = a^2 - W_A^2/4 - 2a^2(p_0+p_2) + aW_A\rho, \tag{229}$$

$$\Phi_k^A(s) = 0, \quad \beta(s + q_1^A) + 4a^2\beta\sqrt{p_\uparrow p_\downarrow}\frac{\cosh(\beta h)\cosh(\beta u) + (-1)^k}{\cosh(\beta u) + (-1)^k\cosh(\beta h)} = 0. \tag{230}$$

The up-oriented null vector is $1/(z_h(\nu)^2 - s)$; for the down orientation reflect $\nu \to -\nu - \omega$. Also $P_A(u) \neq 0$ is required as above. At a real zero the imaginary part of the second condition is $\beta\omega(\mu + a\rho)$.

G.5.3 Equal-spin pairing

On the mirror-odd orbit $n \neq \overline{n}$,

$$\chi_{\mathrm{E}\,\sigma}(\boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}) = \frac{\beta\mathcal{L}_\sigma\big(t_{\sigma\nu_n}\big)}{\mathcal{P}_\sigma\big(t_{\sigma\nu_n}\big)}(\boldsymbol{e}_n - \boldsymbol{e}_{\overline{n}}). \tag{231}$$

It is diagonal in disjoint orbit vectors and has no exceptional points. Its complete divergence set is

$$\omega \neq 0, \quad \mu_\sigma + a r_\sigma = 0, \quad a^2(1 - r_\sigma^2) = -\nu_n(\nu_n + \omega), \quad n \neq \overline{n}. \tag{232}$$

The printed orbit vector is the localized null vector. To see the condition, expand

$$\mathcal{L}_\sigma(t_{\sigma\nu}) = (\mu_\sigma + a r_\sigma)^2 + a^2(1 - r_\sigma^2) + \nu(\nu + \omega) - i\omega(\mu_\sigma + a r_\sigma). \tag{233}$$

At $\omega = 0$ its real part is positive. At $|k| = 1$ the only negative product belongs to the forbidden fixed point, so $|k| \geq 2$ is necessary. Conversely for an allowed negative product $-c$, choose $a = \pm\sqrt{\mu_\sigma^2 + c}$; along $\mu = \mu_\sigma - \sigma h$, r_σ continuously ranges from $+1$ to -1 as σh increases, and takes the required value $-\mu_\sigma/a$. Thus the set is nonempty for either sign of U; it is invariant under the particle–hole map. This construction fixes U through a, rather than asserting existence at arbitrary prescribed U. For example, at $k = 2$ the orbit $\{-1, -2\}$ diverges near $(\beta U, \beta\mu, \beta h) = (10.97247706, 0.56858193, 0.13141807)$ (numerical).

G.5.4 Longitudinal selection, null vectors and exceptional points

Use the simple, nonzero, non-atomic roots and off-axis domain of Eq. 196, and suppose $Q_\uparrow\big(\pm u_j\big) \neq 0$. Define the branch ratio

$$\mathfrak{r}(u) \coloneqq -\frac{2\lambda_L^2(u^2 + q_0)\alpha_\uparrow(u)}{\beta Q_\uparrow(u)}, \quad \Phi_L(s) \coloneqq \frac{\mathfrak{r}(u) - \mathfrak{r}(-u)}{2u}, \quad u^2 = s. \tag{234}$$

At $u = u_j$ this is $\mathfrak{r}_j \coloneqq f_{j\uparrow-}/f_{j\uparrow+}$; write $\mathfrak{r}_j\big(\pm u_j\big) \coloneqq \mathfrak{r}\big(\pm u_j\big)$ using Eq. 193. The polynomial identity

$$Q_\uparrow(u)Q_\uparrow(-u) - 4\lambda_L^4\pi_P\pi_S(u^2 + q_0)^2 = E_8(u)H_\downarrow(u)H_\downarrow(-u) \tag{235}$$

and $\alpha_\uparrow(u)\alpha_\uparrow(-u) = \beta^2\pi_P\pi_S$ give $\mathfrak{r}_j(u)\mathfrak{r}_j(-u) = 1$. Also

$$K_{j\uparrow} = -\beta Q_\uparrow\big(u_j\big)\big[1 + \mathfrak{r}_j\big(u_j\big)\big]. \tag{236}$$

For real $s_j < 0$, $|\mathfrak{r}_j| = 1$; for real $s_j > 0$, $\mathfrak{r}_j$ is real. Thus $\Phi_L\big(s_j\big) = 0$ selects **both** $\mathfrak{r}_j = \pm 1$. The genuine branch is

$$\mathfrak{r}_j = -1 \quad \Longleftrightarrow \quad K_{j\uparrow} = K_{j\downarrow} = 0. \tag{237}$$

On $\mathfrak{r}_j = +1$ the vertex stays finite: $f_{j\uparrow+} - f_{j\uparrow-} = 0$ but $K_{j\uparrow} = -2\beta Q_\uparrow\big(u_j\big) \neq 0$. For negative s_j select $\mathrm{Re}\,\mathfrak{r}_j = -1$, not merely $\Phi_L = 0$. For complex roots impose the full complex equation, generically two real conditions.

At a genuine pole a null vector in spin and frequency is

$$\begin{aligned} g_{j\sigma}(\nu) &= \frac{f_{j\sigma+}}{z_{\mu_\sigma}(\nu)^2 - s_j}, \\ \big(f_{j\uparrow+}, f_{j\downarrow+}\big) &= \big(-\beta Q_\uparrow\big(u_j\big), -\lambda_L\big[H_\uparrow(-u_j)H_\downarrow(-u_j) + \lambda_L^2 d_\pi\big]\alpha_\downarrow^S\big(u_j\big)\big). \end{aligned} \tag{238}$$

The reason is that $K_{j\sigma} = 0$ makes the mode in Eq. 196 proportional to this vector; its polynomial image under $P_L \chi_L$ vanishes when the thermal denominators do, whereas the nonzero residue numerator makes the inverse pole genuine.

At a simple null direction the exceptional-point test is

$$\boldsymbol{g}_j^T \boldsymbol{g}_j = \sum_{\sigma \in \{\uparrow,\downarrow\}} f_{j\sigma+}^2 T_{(\mu_\sigma)'}(s_j) = 0, \tag{239}$$

using T_c of Eq. 226. Symmetry of the coupled susceptibility gives the same necessary self-orthogonality condition as in Eq. 216; finite-box diagnosis of defectiveness remains numerical. The longitudinal spectrum is centrohermitian at every transfer under the joint mirror on both spins. In the slice $\beta h = 0.5$ the first static loop has an exceptional divergence near $(\beta\mu, \beta U) = (0.64162, 4.78028)$ (numerical). Very near the omitted atomic root, $1 + \mathfrak{r}_j$ can be tiny without crossing zero; double precision is not a reliable branch test there.

G.5.5 Axis reductions and numerical checks

At $h = 0$, $E_8 = e_{\mathrm{d}} e_{\mathrm{m}}$; the longitudinal spectrum is the union of the zero-field density and magnetic spectra, with spin null vectors (g, g) and $(g, -g)$ respectively. The ratio test reduces to their $K_j = 0$ selection. The transverse sector is magnetic. The opposite-spin pair has the singlet spectrum on $\mathcal{M}_+$ and the triplet spectrum on $\mathcal{M}_-$, with the physical factors 2 and -2 respectively ($\chi_A \mathcal{M}_+ = 2\chi_{\mathrm{s}} \mathcal{M}_+$, $\chi_A \mathcal{M}_- = -2\chi_{\mathrm{t}} \mathcal{M}_-$); its vertex reduction is Eq. 198. Equal-spin pairing is the triplet problem on the allowed subspace with the factor -2 in χ.

At $\mu = 0$ the orthogonal longitudinal similarity of Eq. 200 preserves eigenvalues, nullity and bilinear self-overlap: in the charge/S_z basis a source density null vector maps to $(\mathcal{M}_- g, \mathcal{M}_+ g)$ and a magnetic one to $(\mathcal{M}_+ g, \mathcal{M}_- g)$. The opposite-spin pair is the magnetic source $(-U, \beta, h)$; the transverse sector has the source singlet/triplet split just described. In its mirror form the global pole is $\hat{D} = 0$, and the localized poles obey $\beta H_T(z_n) = 4a^2 \beta p_0$ for $n \neq \overline{n}$; the opposite sign is removable. At $\mu = h = 0$ recover the localized and global conditions already listed in this appendix.

On either $h = 0$ or $\mu = 0$, equal-spin pairing has no divergences, and at $\mu = 0$, $h \neq 0$ there are no nonstatic transverse divergences. These axis exclusions follow by substituting the zero-field thermal weights in the pair conditions (the zero-field singlet/triplet nonstatic exclusion); they do not extend to simultaneous doping and field. On $\mu = \pm h$ the longitudinal ratio theorem does not apply: use the continuous extension, distinguishing genuine axis lattice poles from removable limits as in Appendix F.6.5. No off-axis lattice-root exclusion is claimed.

`doped-gamma/check_field.jl` transcribes the displayed field formulas independently of the package's vertex code at 256 bits, tests both orientations and the axis reductions, and tests predicted divergences against the package susceptibility on growing boxes, including overlaps with the printed null vectors. Equal-spin boxes are restricted to $\mathcal{M}_-$ before taking eigenvalues; their structural even-subspace zeros are not divergences. At 256 bits the worst normalized vertex difference over 2882 comparisons is 6.4×10^{-63}; the branch-ratio identity differs from one by at most 2.3×10^{-60}. At the tested nonlocalized divergences, the largest smallest-box eigenvalue falls from 3.7×10^{-8} at 24 frequencies to 3.8×10^{-11} at 96, with overlap deficit at most 3.2×10^{-8}; the restricted equal-spin eigenvalues are below 1.5×10^{-18}. The transverse and longitudinal exceptional pairs shrink from about 1.1×10^{-6} and 8.4×10^{-7} at 48 frequencies to 3.4×10^{-8} and 2.6×10^{-8} at 192 (per spin in the longitudinal case). These measured box convergence, overlaps and exceptional-point checks are numerical evidence, not error bounds or proofs of the unresolved extensions.

H List of symbols

Symbols introduced inside a proof, in Appendix F or in Appendix G are local to it and not listed, except those of the proof of Theorem 4 that Appendix B uses and the field symbols shared by Appendices F and G.

Setup and input.

Symbol	Meaning	Defined in
U, μ, β, a	interaction, chemical potential (zero at half filling), inverse temperature; $a = U/2$	Eq. 3
p_0, p_1, p_2	Boltzmann weights of $\|0\rangle$, of each singly occupied state, of $\|\uparrow\downarrow\rangle$	Eq. 4
ρ, h, $\overline{c}$	$p_2 - p_0$ (filling), $\beta\sqrt{p_0 p_2}$, $\beta(p_0 + p_2)/2$	Eq. 5
ν_n, Ω_ℓ, ω_k	fermionic and bosonic Matsubara frequencies; the fixed transfer $\omega_k = \Omega_k$	Eq. 8
x_n, w, J	$\mu + i\nu_n$, $i\omega_k$, $J(x) = 1/(x^2 - a^2)$	Eq. 9
G	propagator, as a function of ν and as the rational function $G(x)$	Eq. 11
$\chi_{\sigma\sigma'}$, $\chi_{\mathrm{pp},\sigma\sigma'}$	generalized susceptibility and its particle–particle form	Eq. 12
χ_r, $\chi_{0,r}$, B, B_n, $\boldsymbol{B}$	channels $r = \mathrm{d}, \mathrm{m}, \mathrm{s}, \mathrm{t}$, bare susceptibility, bubble	Eq. 15
Γ_r, η_r	channel-irreducible vertex, sign of the bubble term	Eq. 16
z_n, $\overline{n}$	centered lattice, mirror index	Eq. 17
$\mathcal{M}$, PV	mirror matrix; principal value along symmetric boxes	Section 2.4
$\mathcal{N}$, c_ℓ, Π, R_{reg}	the part of $\chi_{\uparrow\downarrow}$ with the pair coefficient c_ℓ, and its regular part	Eq. 20
C, $\mathcal{C}$	Cauchy matrix; its action on functions on the lattice	Eq. 21, Lemma 15
$Q_\pm$, q_0	$Q_\pm(z) = (z \pm a)^2 - w^2/4$, $q_0 = a^2 - w^2/4$	Eq. 22
S_k, $S_\pm$	thermal sum $\mathrm{PV}\sum_n (z_n - x)^{-1}$; its values at the zeros of $Q_\pm$	Eq. 28, Lemma 8

Singlet and triplet.

Symbol	Meaning	Defined in
b, t_n	$2\mu - w$; $x_n(b - x_n) = t_{\overline{n}}$	Eq. 29
$\mathcal{P}$, $\mathcal{G}$, A_{s}	polynomials of the particle–particle channels	Eq. 30
$\mathcal{T}_k$, ξ	particle–particle sum $\sum_n (t_n - t)^{-1}$	Eq. 30
K_{s}	thermal denominator of the singlet	Theorem 3
$t_\pm$	zeros of $\mathcal{P}$	Lemma 10
$d_\pm$, c	$A_{\mathrm{s}} + t_\pm$; the pair coefficient c_{-k}	Eq. 62
$\mathcal{M}_\pm$, L_{s}, L_{t}, $g_\pm$, $\gamma_\pm$	mirror projectors; eigenvalues and rank-two part of χ_{s}, χ_{t}	Eq. 67
$\chi_{\mathrm{s,\,reg}}$	rank-two part of χ_{s}	Eq. 69

Density and magnetic.

Symbol	Meaning	Defined in
ε_r	channel sign, $\varepsilon_{\mathrm{d}} = +1$, $\varepsilon_{\mathrm{m}} = -1$	Section 4.2
H, q_1, λ, P, G_2, e	polynomials of the particle–hole channels	Eq. 35, Eq. 36
s_j, u_j	roots of e in z^2; either square root	Eq. 36
α_k	thermal function $\rho S_k - \overline{c}$	Eq. 38
F, g_j, $\mathbb{D}$	ingredients of the dense and separable terms	Eq. 39
K_j, Z_j, γ_j	thermal denominator, numerator and weight of the separable term j	Eq. 39
δ, ζ_r, N_2, N_0, N	finite-rank part of $P\chi_r$	Eq. 103
$\mathcal{I}$, $\overline{\mathcal{I}}$	rank-two sums	Eq. 110
y_v, D_v, E_v, F_j	column of χ_r^{-1} and its coefficients	proof of Theorem 4
φ_j, α_j	pole pair (mode) at the root u_j	Eq. 122
W, W_0, W_2	remainder polynomials of Step 1	Eq. 117, (A7)
$K(u)$, $Z(u)$, $Z(s)$	K_j, Z_j as functions of the root; even part of Z	Eq. 133, Proposition 6

Half filling and removable singularities.

Symbol	Meaning	Defined in
$\hat{s}$, $\hat{s}'$	roots of e in z^2 at $\mu = 0$	Eq. 41
$\hat{g}$, $\hat{D}$, $\hat{u}$	separable factor, thermal denominator, $\hat{u}^2 = \hat{s}$	Eq. 42
O, c	mirror orbit $\{c, \overline{c}\}$ with $\lvert O\rvert$ elements	Section 4.3
T', K'	$\mathcal{T}_k(-A_{\mathrm{s}})$ and K_{s} without the orbit	Eq. 46
$\hat{T}'$, $\hat{D}'$	$\sum_l \hat{g}(z_l)$ and $\hat{D}$ without the orbit	Eq. 49

Doping and field (local to the field parts of Appendices F–G).

Symbol	Meaning	Defined in
h, Z_f, $p_\uparrow$, $p_\downarrow$	physical field (splitting $2h$), normalization, spin-resolved weights	Appendix F.6.1
κ, m_f, μ_σ, r_σ	thermal constant $\beta\sqrt{p_0 p_2}$ (called h outside the field parts), magnetization, spin chemical potential, $2\langle n_{-\sigma}\rangle - 1$	Appendix F.6, Eq. 172
L, T σ, A σ, E σ, b_X	coupled longitudinal, transverse, oriented opposite-spin and equal-spin pairs; physical bubbles	Appendix F.6.1, Eq. 174
z_c, S_c, W	centered physical frequency, thermal sum, complex transfer $i\omega - 2h$	Eq. 175, Eq. 179
H_T, P_T, e_T, q_1	transverse polynomials; s_j, u_j are their root pairs	Eq. 179
$\mathcal{P}_\sigma$, $\mathcal{G}_\sigma$, $\mathcal{L}_\sigma$, b_σ, $t_{\sigma\nu}$	equal-spin pair polynomials and frequency variables	Appendix F.6.3
λ_L, π_P, π_S, D_t, c_t, d_π	longitudinal interaction and weight combinations	Eq. 189
H_t, P_L, Q_t, E_8, $E'_{8,j}$	longitudinal polynomials; derivative in s at a root	Eq. 189, Appendix F.6.4
α_t, α_t^P, α_t^S, $f_{j\sigma\pm}$	thermal factors and pole-pair mode coefficients	Eq. 193
K_{jt}, $\mathfrak{c}_{jt}$, Ξ_j, Π_j	longitudinal thermal denominator and separable coefficient	Eq. 193, Eq. 195
ε_σ, R_σ, $\mathcal{S}$	spin sign, down-spin frequency reflection, orthogonal Shiba matrix	Eq. 200
D_k, R_k, Φ_k^T, K_e^T, ℓ_T, Φ_k^A	transverse spectral and divergence functions; their pair images	Eq. 221, Eq. 230
$\mathfrak{r}_j$, Φ_L, T_c	longitudinal branch ratio, scalar selection function, lattice sum for self-overlap	Eq. 234, Eq. 226