Notes on 2D T-matrix for elastic waves

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1 Preliminaries

The governing equations for elastic waves in linear isotropic media are

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \omega^2\mathbf{u} = 0 \tag{1.1}$$

where $\mathbf{u}$ is the displacement field, λ and μ are the Lamé parameters, and ω is the angular frequency. We can decompose the displacement field into using the Helmholtz decomposition:

$$\mathbf{u} = \nabla\phi + \nabla\times\boldsymbol{\psi} \tag{1.2}$$

where ϕ is the pressure potential and ψ is the shear potential for the waves, due to the problem being 2D we can write $\boldsymbol{\psi} = \psi\boldsymbol{e}_z$ with $\nabla\cdot\boldsymbol{\psi} = 0$, where $\boldsymbol{e}_z$ is the unit vector in the z-direction. Substituting this decomposition into the governing equations, we can derive separate equations for pressure and shear potentials. The pressure and shear potentials ϕ and ψ satisfies the Helmholtz equation:

$$\nabla^2\phi + k_p^2\phi = 0 \quad \text{and} \quad \nabla^2\psi + k_s^2\psi = 0. \tag{1.3}$$

where $k_p = \omega/c_p$ is the wavenumber for the pressure waves, and c_p is the speed of pressure waves. The wavenumber for the shear waves is $k_s = \omega/c_s$, where c_s is the speed of shear waves. Under the isotropic and homogeneous assumptions,the stress tensor σ is related to the displacement field $\mathbf{u}$ by:

$$\boldsymbol{\sigma} = \lambda(\nabla\cdot\boldsymbol{u})\mathbf{I} + \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}), \tag{1.4}$$

We wish to evaluate traction boundary conditions on a circle. So, the components σ_{rr} and $\sigma_{r\theta}$ can be rewritten in terms of the potentials in cylindrical coordinates as

$$\begin{aligned}
\sigma_{rr} &= -\lambda k_p^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\right)\\
\sigma_{r\theta} &= -\mu k_s^2\psi - 2\mu\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right)
\end{aligned} \tag{1.5}$$

2 Waves clapping the crack (single scatterer)

Consider a circular crack of radius a centered at the origin. The boundary condition for this is traction zero on the boundary of the crack which is mathematically written as $\boldsymbol{\tau} = \sigma \cdot \boldsymbol{n}|_{r=a} = 0$ where $\boldsymbol{n} = -\boldsymbol{e}_r$ is the outward normal vector. So the boundary conditions can be written as

$$\boldsymbol{\tau}(a,\theta) = \boldsymbol{\tau}_{\text{inc}} + \boldsymbol{\tau}_{\text{scat}} = 0, \quad \text{where} \quad \boldsymbol{\tau} = -\begin{bmatrix}\sigma_{rr}(a,\theta)\\ \sigma_{r\theta}(a,\theta)\end{bmatrix} = \begin{bmatrix}p_{\text{inc}}(a,\theta) + p_{\text{scat}}(a,\theta)\\ s_{\text{inc}}(a,\theta) + s_{\text{scat}}(a,\theta)\end{bmatrix} \tag{2.1}$$

Now we connect the potentials with the corresponding pressure and shear tractions. We can in general write the potentials as a sum of incident and scattered waves, where the incident wave, can be genrally written as

$$\phi^{\text{inc}}(r,\theta) = \sum_{n=-\infty}^{\infty} f_n^p J_n(k_p r) e^{in\theta} \quad \text{and} \quad \psi^{\text{inc}}(r,\theta) \quad = \sum_{n=-\infty}^{\infty} f_n^s J_n(k_s r) e^{in\theta} \tag{2.2}$$

and the scattered wave can be written as

$$\phi^{\text{scat}}(r,\theta) = \sum_{n=-\infty}^{\infty} g_n^p H_n^{(1)}(k_p r) e^{in\theta} \quad \text{and} \quad \psi^{\text{scat}}(r,\theta) \quad = \sum_{n=-\infty}^{\infty} g_n^s H_n^{(1)}(k_s r) e^{in\theta} \tag{2.3}$$

where J_n is the Bessel function, and $H_n^{(1)}$ is the Hankel function of the first kind. The T matrix is the matrix that connects the coefficients of the incident waves with the coefficients of the scatttered fields, so we can write

$$\boldsymbol{g_n} = \mathbf{T}_{nm}\boldsymbol{f}_m \quad \text{where} \quad \boldsymbol{g_n} = \begin{bmatrix} g_n^p \\ g_n^s \end{bmatrix} \quad \text{and} \quad \boldsymbol{f}_m = \begin{bmatrix} f_m^p \\ f_m^s \end{bmatrix} \tag{2.4}$$

Since we already did similar calculations to calculate $\mathbf{M}^{\text{for}}$ and $\mathbf{M}^{\text{inv}}$ in [1], we can use it to calculate the T matrix which is the matrix that connects the coefficients of the incident waves with the coefficients of the scatttered fields. So first consider the following functions

$$
\begin{aligned}
P_1^{(n)}(r) &= -\frac{\rho}{r^2}\left(2c_s^2 k_p r \mathrm{J}_{n-1}(k_p r) + \left(r^2\omega^2 - 2c_s^2 n(n+1)\right)\mathrm{J}_n(k_p r)\right)\\
P_2^{(n)}(r) &= -\frac{\rho}{r^2}\left(2c_s^2 k_p r \mathrm{H}_{n-1}^{(1)}(k_p r) + \left(r^2\omega^2 - 2c_s^2 n(n+1)\right)\mathrm{H}_n^{(1)}(k_p r)\right)\\
P_3^{(n)}(r) &= \frac{\mathrm{i}\rho c_s^2 n}{r^2}\left(k_s r \mathrm{J}_{n-1}(k_s r) - 2\mathrm{J}_n(k_s r) - k_s r \mathrm{J}_{n+1}(k_s r)\right)\\
P_4^{(n)}(r) &= \frac{\mathrm{i}\rho c_s^2 n}{r^2}\left(k_s r \mathrm{H}_{n-1}^{(1)}(k_s r) - 2\mathrm{H}_n^{(1)}(k_s r) - k_s r \mathrm{H}_{n+1}^{(1)}(k_s r)\right)\\
S_1^{(n)}(r) &= \frac{\mathrm{i}\rho c_s^2 n}{r^2}\left(k_p r \mathrm{J}_{n-1}(k_p r) - 2\mathrm{J}_n(k_p r) - k_p r \mathrm{J}_{n+1}(k_p r)\right)\\
S_2^{(n)}(r) &= \frac{\mathrm{i}\rho c_s^2 n}{r^2}\left(k_p r \mathrm{H}_{n-1}^{(1)}(k_p r) - 2\mathrm{H}_n^{(1)}(k_p r) - k_p r \mathrm{H}_{n+1}^{(1)}(k_p r)\right)\\
S_3^{(n)}(r) &= \frac{\rho}{r^2}\left(2c_s^2 k_s r \mathrm{J}_{n-1}(k_s r) + \left(r^2\omega^2 - 2c_s^2 n(n+1)\right)\mathrm{J}_n(k_s r)\right)\\
S_4^{(n)}(r) &= \frac{\rho}{r^2}\left(2c_s^2 k_s r \mathrm{H}_{n-1}^{(1)}(k_s r) + \left(r^2\omega^2 - 2c_s^2 n(n+1)\right)\mathrm{H}_n^{(1)}(k_s r)\right).
\end{aligned}
$$

Using that to calculate the incident pressures we obtain

$$
-p_{\mathrm{inc}}^n(r) = P_1^{(n)}(r) f_n^p + P_3^{(n)}(r) f_n^s \quad \text{and} \quad -s_{\mathrm{inc}}^n(r) = S_1^{(n)}(r) f_n^p + S_3^{(n)}(r) f_n^s \tag{2.5}
$$

and the scattered fields

$$
-p_{\mathrm{scat}}^n(r) = P_2^{(n)}(r) g_n^p + P_4^{(n)}(r) g_n^s \quad \text{and} \quad -s_{\mathrm{scat}}^n(r) = S_2^{(n)}(r) g_n^p + S_4^{(n)}(r) g_n^s \tag{2.6}
$$

Now we can write the boundary conditions in matrix form as

$$
\begin{bmatrix} P_2^{(n)}(a) & P_4^{(n)}(a) \\ S_2^{(n)}(a) & S_4^{(n)}(a) \end{bmatrix} \begin{bmatrix} g_n^p \\ g_n^s \end{bmatrix} = - \begin{bmatrix} P_1^{(n)}(a) & P_3^{(n)}(a) \\ S_1^{(n)}(a) & S_3^{(n)}(a) \end{bmatrix} \begin{bmatrix} f_n^p \\ f_n^s \end{bmatrix} \tag{2.7}
$$

The T matrix is the matrix that connects the coefficients of the incident waves with the coefficients of the scatttered fields, so we can write

$$
\begin{bmatrix} g_n^p \\ g_n^s \end{bmatrix} = - \begin{bmatrix} P_2^{(n)}(a) & P_4^{(n)}(a) \\ S_2^{(n)}(a) & S_4^{(n)}(a) \end{bmatrix}^{-1} \begin{bmatrix} P_1^{(n)}(a) & P_3^{(n)}(a) \\ S_1^{(n)}(a) & S_3^{(n)}(a) \end{bmatrix} \begin{bmatrix} f_n^p \\ f_n^s \end{bmatrix} \tag{2.8}
$$

So the T matrix do not mix the modes $\mathbf{T}_{nm} = -\delta_{nm}\mathbf{Z}_m$

$$
\mathbf{Z}_n = \begin{bmatrix} P_2^{(n)}(a) & P_4^{(n)}(a) \\ S_2^{(n)}(a) & S_4^{(n)}(a) \end{bmatrix}^{-1} \begin{bmatrix} P_1^{(n)}(a) & P_3^{(n)}(a) \\ S_1^{(n)}(a) & S_3^{(n)}(a) \end{bmatrix}. \tag{2.9}
$$

References

[1] Jessica J. Kent et al. "Elastic waves in bearing raceways: the forward and inverse problem". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 481.2254 (2025), p. 20240972. URL: https://royalsocietypublishing.org/doi/10.1098/rspa.2024.0972.