s/eLORETA model-driven & (new) data-driven inverse solutions for EEG/MEG

Marco Congedo,

GIPSA-lab,

CNRS, Grenoble Alpes University, Grenoble-INP

M. Congedo (2013) *EEG Source Analysis*. HDR presented at Doctoral School EDISCE, University of Grenoble.

Model-Driven Linear Distributed Inverse Solutions
A short History

1984 (Hämäläinen and Ilmoniemi), 1987 (Sarvas)
Minimum Norm: no depth localization, huge localization error

1984→ (many authors)
Weighted Minimum Norm

1994 (Pascual-Marqui)
LORETA: low localization error

1997 (Van Veen et al.)
LCMVB: localization error, not a genuine solution

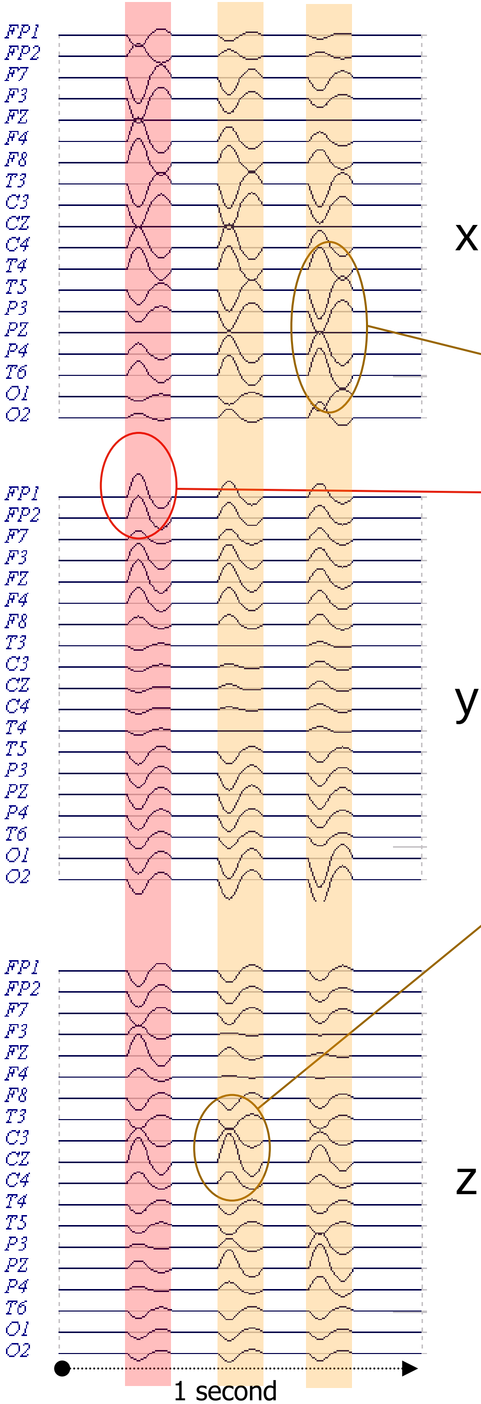
2002 (Pascual-Marqui)
sLORETA -> zero localization error, not a genuine solution

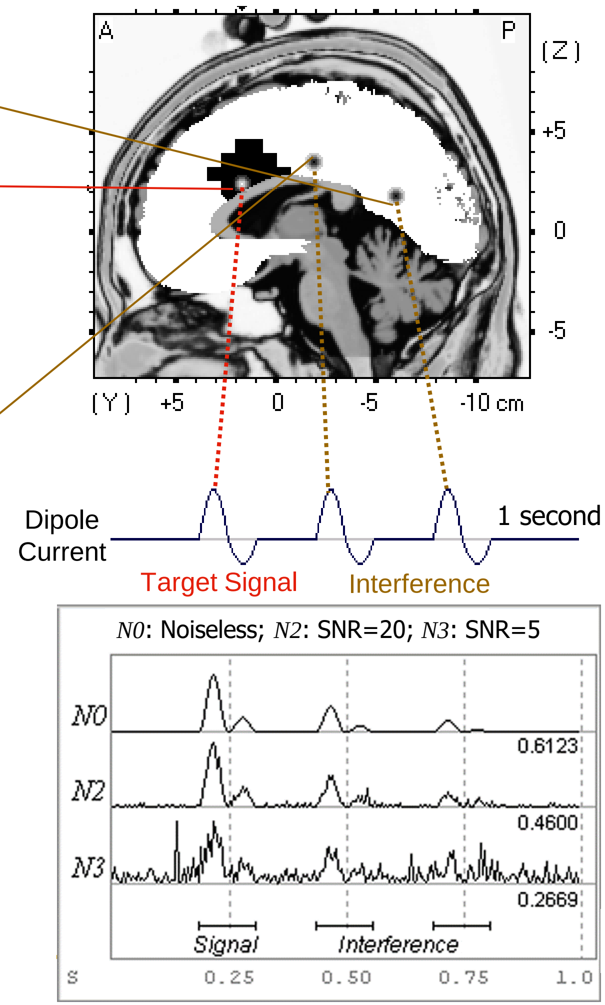
2007 (Pascual-Marqui)
eLORETA -> zero localization error and a genuine solution

All the model driven methods results in smooth, low spatial resolution reconstruction

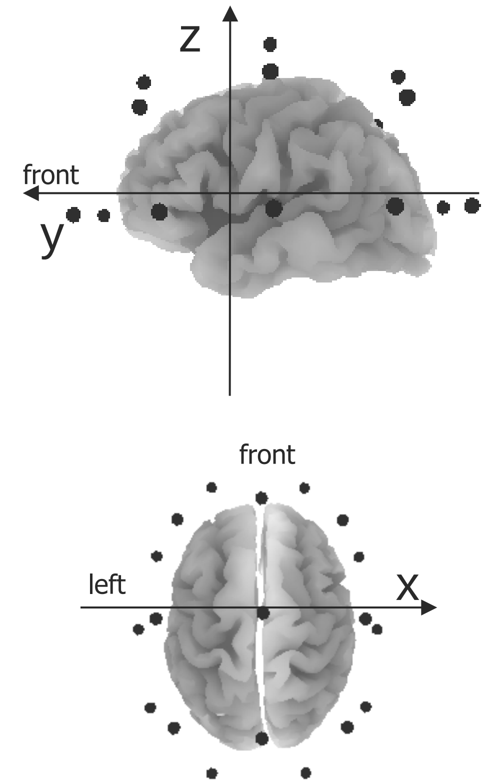
Pascual-Marqui R.D. et al. (2018) Comparing EEG/MEG neuroimaging methods based on localization error, false positive activity, and false positive connectivity. bioRxiv.

Spatial Resolution
(19 electrodes)





sLORETA Current Density



Congedo (2006), IEEE TBE

Notation, Nomenclature and Problem Statement

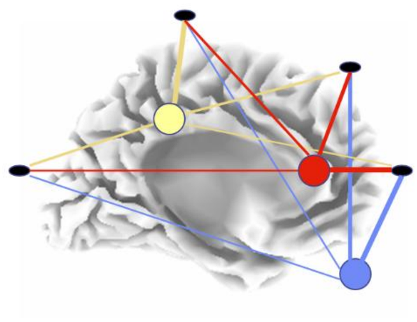
N sensors $\quad \boldsymbol{x}(t) \leftarrow \boldsymbol{H}\boldsymbol{x}(t) \in \mathbb{R}^{N}$ is the CAR Sensor Measurement Vector

$\boldsymbol{H} = \boldsymbol{I} - \tfrac{1}{N}\boldsymbol{1}\boldsymbol{1}^{T}$ is the centering matrix

Q voxels $\quad \boldsymbol{j}(t) \in \mathbb{R}^{3Q}$ Current Vector

$\boldsymbol{x}(t) = \boldsymbol{K}\boldsymbol{j}(t)$ *Forward* problem

$\boldsymbol{j}(t) = \boldsymbol{T}^{T}\boldsymbol{x}(t)$ *Inverse* problem



$\boldsymbol{K} \in \mathbb{R}^{N\cdot 3Q}; \quad \boldsymbol{K} = \left[\boldsymbol{K}_1\boldsymbol{K}_2 \ldots \boldsymbol{K}_Q\right]; \quad \boldsymbol{K}_q = \left[\boldsymbol{k}_x\boldsymbol{k}_y\boldsymbol{k}_z\right]_q \in \mathbb{R}^{N\cdot 3}$ Leadfield Matrix

$\boldsymbol{T}^{T} \in \mathbb{R}^{3Q\cdot N}; \quad \boldsymbol{T} = \left[\boldsymbol{T}_1\boldsymbol{T}_2 \ldots \boldsymbol{T}_Q\right]; \quad \boldsymbol{T}_q = \left[\boldsymbol{t}_x\boldsymbol{t}_y\boldsymbol{t}_z\right]_q \in \mathbb{R}^{N\cdot 3}$ Transfer Matrix

Notation, Nomenclature and Problem Statement

Given a leadfield matrix $\boldsymbol{K}$ the inverse problem has infinite solutions $\boldsymbol{T}$.

A genuine solution should reproduce the measurement such that $\boldsymbol{x}(t) = \boldsymbol{K}\boldsymbol{T}^T\boldsymbol{x}(t) \rightarrow \boldsymbol{K}\boldsymbol{T}^T = \boldsymbol{H}$

and should feature zero-localization error in psf $\rightarrow$ $\boldsymbol{T}^T\boldsymbol{K}$ is 'block-diagonally dominant'

$$\boldsymbol{j}_q = \boldsymbol{T}_q^T \boldsymbol{x} \in \mathbb{R}^3$$

Single-Voxel *Current*

$$\gamma_q = \left\| \boldsymbol{j}_q \right\|^2 = \boldsymbol{x}^T \boldsymbol{\Xi}_q \boldsymbol{x}, \qquad \boldsymbol{\Xi}_q = \boldsymbol{T}_q \boldsymbol{T}_q^T \in \mathbb{R}^{N \cdot N}$$

Single-Voxel *Current Density*

$$\gamma_\Omega = \sum_{q \in \Omega} \boldsymbol{x}^T \boldsymbol{\Xi}_q \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{\Xi}_\Omega \boldsymbol{x}, \quad \boldsymbol{\Xi}_\Omega = \sum_{q \in \Omega} \boldsymbol{T}_q \boldsymbol{T}_q^T \in \mathbb{R}^{N \cdot N}$$

ROI *Current Density*

For Beamforming ROI Current density Estimation see Congedo (2006), IEEE TBE

Covariance (Data) and LeadField (Model)

$\boldsymbol{C} = E(\boldsymbol{x}\boldsymbol{x}^T)$ Sensor Covariance Matrix

$\boldsymbol{x}(t) = \boldsymbol{K}\boldsymbol{j}(t)$ Forward Problem

$\boldsymbol{C} = \boldsymbol{k}_{xq}\boldsymbol{k}_{xq}^T$ for 1 dipole x-oriented at voxel q and amplitude 1

$\boldsymbol{C} = \boldsymbol{K}_q\boldsymbol{\theta}_q\boldsymbol{\theta}_q^T\boldsymbol{K}_q^T$ for 1 dipole oriented as $\boldsymbol{\theta}_q = [x, y, z]$ at voxel q and amplitude 1

$\boldsymbol{C} = \boldsymbol{K}_q\boldsymbol{\theta}_q\sigma_q^2\boldsymbol{\theta}_q^T\boldsymbol{K}_q^T$ for 1 dipole oriented as $\boldsymbol{\theta}_q$ and amplitude σ_q at voxel q

$\boldsymbol{C} = \boldsymbol{K}_q\boldsymbol{\theta}_q\sigma_q^2\boldsymbol{\theta}_q^T\boldsymbol{K}_q^T + \boldsymbol{K}_j\boldsymbol{\theta}_j\sigma_j^2\boldsymbol{\theta}_j^T\boldsymbol{K}_j^T$ for 2 dipole oriented as $\boldsymbol{\theta}_p$ and $\boldsymbol{\theta}_q$ and amplitude σ_p and σ_q at voxel p and q

$\boldsymbol{K}\boldsymbol{K}^T$ Gram Matrix

Weighted Regularized Minimum Norm Solution

$\boldsymbol{j} = \boldsymbol{T}^T \boldsymbol{x}$ *Inverse* problem

$\boldsymbol{T}^T = \boldsymbol{K}^T \left(\boldsymbol{K}\boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^{-1}, \ \alpha \geq 0$ → Minimum Norm

if $\alpha = 0$ *then* $\boldsymbol{T}^T = \boldsymbol{K}^+$

$\boldsymbol{T}^T = \boldsymbol{\Theta}^{-1} \boldsymbol{K}^T \left(\boldsymbol{K}\boldsymbol{\Theta}^{-1}\boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^{-1},$ → Weighted Minimum Norm

(e.g., LORETA, eLORETA, but NOT sLORETA)

with $\boldsymbol{\Theta} \in \mathbb{R}^{3Q \cdot 3Q}$ symmetric

sLORETA and eLORETA

sLORETA family (NOT a genuine solution)

$$\boldsymbol{T}_q^T = \left[\left(\boldsymbol{K}_q^T \boldsymbol{Z} \boldsymbol{K}_q \right)^{-1/2} \boldsymbol{K}_q^T \boldsymbol{Z} \right]; \qquad \boldsymbol{Z} \in \mathbb{R}^{N \cdot N} \ \textit{symmetric}$$

$$\boldsymbol{Z} = \left(\boldsymbol{K}\boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^{-1}$$ → sLORETA (model driven, maximum entropy, no a-priori)

$$\boldsymbol{Z} = \left(\boldsymbol{C} + \alpha \boldsymbol{H} \right)^{-1}$$ → sLORETA (data driven)

eLORETA (a genuine solution)

$$\boldsymbol{T}_q^T = \boldsymbol{\Theta}_q^{-1} \boldsymbol{K}_q^T \left(\boldsymbol{K} \boldsymbol{\Theta}^{-1} \boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^{-1}, \ \boldsymbol{\Theta} \in \mathbb{R}^{3Q \cdot 3Q} \text{ symmetric and block-diagonal,}$$

$$\boldsymbol{\Theta}_q^{-1} = \left[\boldsymbol{K}_q^T \left(\boldsymbol{K} \boldsymbol{\Theta}^{-1} \boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^{-1} \boldsymbol{K}_q \right]^{1/2}$$

$\boldsymbol{K}\boldsymbol{K}^T$ Gram Matrix

$\boldsymbol{K}\boldsymbol{\Theta}^{-1}\boldsymbol{K}^T$ Weighted Gram Matrix

eLORETA: another view

$$\boldsymbol{\Theta}_q^{-1} = \left[\boldsymbol{K}_q^T \left(\boldsymbol{K}\boldsymbol{\Theta}^{-1}\boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^+ \boldsymbol{K}_q \right]^{1/2}$$

is the solution to the optimization problem:

$$\min_{\boldsymbol{\Theta}} \left\| \boldsymbol{I} - \left[\boldsymbol{\Theta}^{-1}\boldsymbol{K}^T \left(\boldsymbol{K}\boldsymbol{\Theta}^{-1}\boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^+ \boldsymbol{K}\boldsymbol{\Theta}^{-1} \right] \right\|^2,$$

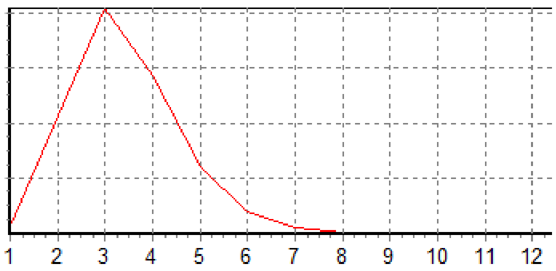
which satisfies the set of matrix equations

$$\boldsymbol{\Theta}_q^2 = \boldsymbol{K}_q^T \left(\boldsymbol{K}\boldsymbol{\Theta}^{-1}\boldsymbol{K}^T + \alpha \boldsymbol{H} \right)^+ \boldsymbol{K}_q, \text{ for all } q : 1 \ldots Q$$

In practice: within-voxels and between-voxels uncorrelation criterium!

eLORETA algorithm

Example Convergence (N=19)



Initialize $\boldsymbol{\Theta} = \boldsymbol{I} \in \mathbb{R}^{3Q \cdot 3Q}$

Repeat

$$\boldsymbol{\Pi} = \left(\boldsymbol{K} \boldsymbol{\Theta}^{-1} \boldsymbol{K}^{T} + \alpha \boldsymbol{H} \right)^{+} \in \mathbb{R}^{N \cdot N}$$

for each voxel q do $\boldsymbol{\Theta}_q^{-1} = \left(\boldsymbol{K}_q^{T} \boldsymbol{\Pi} \boldsymbol{K}_q \right)^{-1/2} \in \mathbb{R}^{3 \cdot 3}$

$\rightarrow$ eLORETA (model driven)

Until Convergence (small changes in $\boldsymbol{\Theta}$)

Trick: $\boldsymbol{K} \boldsymbol{\Theta}^{-1} \boldsymbol{K}^{T} = \sum_q \boldsymbol{K}_q \boldsymbol{\Theta}_q^{-1} \boldsymbol{K}_q^{T}$

$$\boldsymbol{\Pi} = \left(\boldsymbol{C} + \alpha \boldsymbol{H} \right)^{+} \in \mathbb{R}^{N \cdot N}$$

for each voxel q do $\boldsymbol{\Theta}_q^{-1} = \left(\boldsymbol{K}_q^{T} \boldsymbol{\Pi} \boldsymbol{K}_q \right)^{-1/2} \in \mathbb{R}^{3 \cdot 3}$

$\rightarrow$ eLORETA (data driven) ?

Finally compute for each voxel q $\quad \boldsymbol{T}_q^{T} = \boldsymbol{\Theta}_q^{-1} \boldsymbol{K}_q^{T} \left(\boldsymbol{K} \boldsymbol{\Theta}^{-1} \boldsymbol{K}^{T} + \alpha \boldsymbol{H} \right)^{+}$

Interesting Similarities

Vector Type

$$\boldsymbol{T}_q^T = \left[\left(\boldsymbol{K}_q^T \boldsymbol{C}^+ \boldsymbol{K}_q \right)^{-1/2} \boldsymbol{K}_q^T \boldsymbol{C}^+ \right]$$

→ sLORETA (data driven)

$$\boldsymbol{T}_q^T = \left[\left(\boldsymbol{K}_q^T \boldsymbol{C}^+ \boldsymbol{K}_q \right)^{-1} \boldsymbol{K}_q^T \boldsymbol{C}^+ \right]$$

→ LCMV Beamforming

(must be normalized somehow: Sekihara and Nagarajan, 2008)

Scalar Type

$$\boldsymbol{T}_q^T = \frac{\boldsymbol{k}_q^T \boldsymbol{C}^+}{\sqrt{\boldsymbol{k}_q^T \boldsymbol{C}^+ \boldsymbol{k}_q}}$$

→ sLORETA (data driven)

$$\boldsymbol{T}_q^T = \frac{\boldsymbol{k}_q^T \boldsymbol{C}^+}{\boldsymbol{k}_q^T \boldsymbol{C}^+ \boldsymbol{k}_q}$$

→ LCMV Beamforming

(must be normalized somehow)

Relevant Performance Indeces

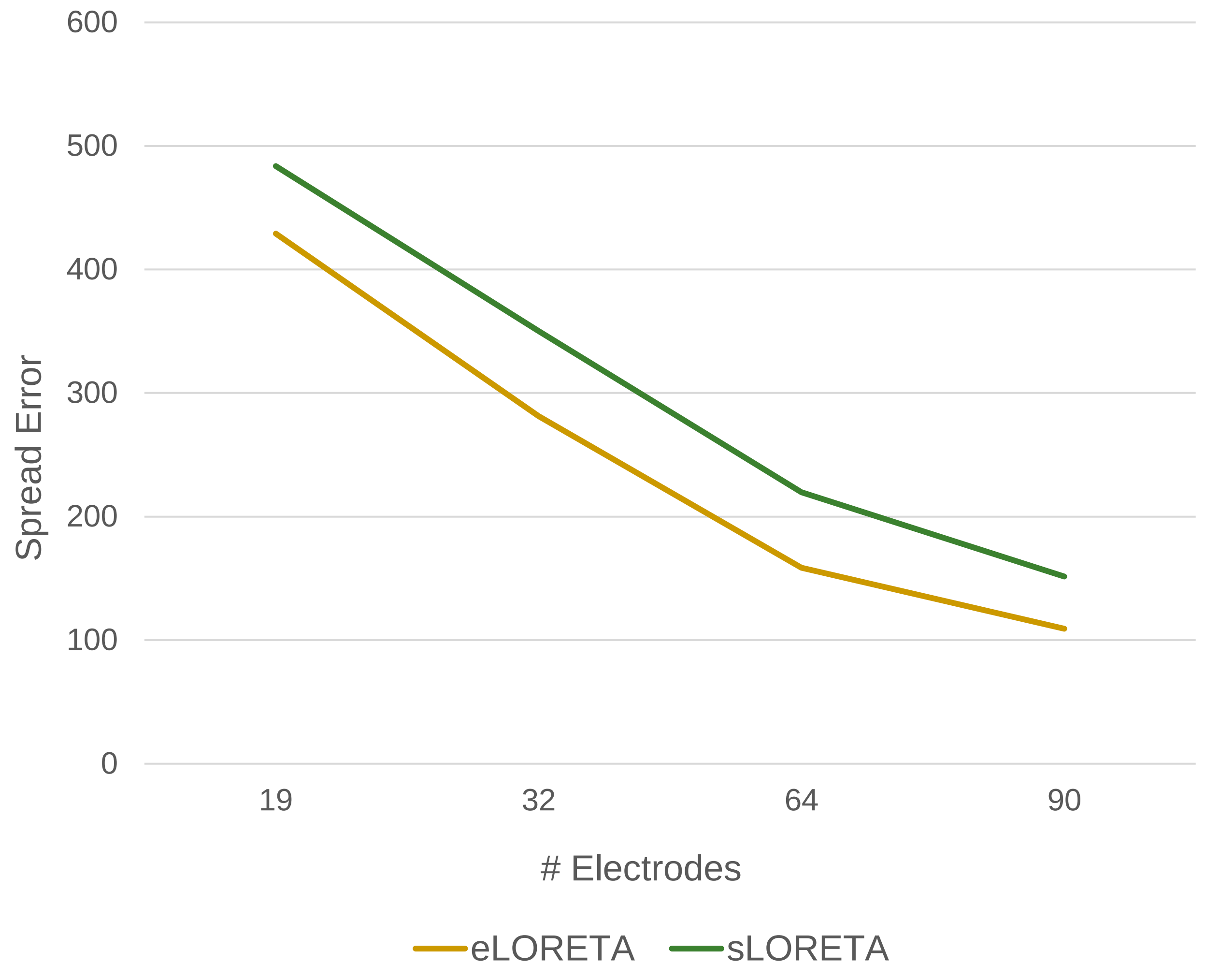
- Spread Error:

Average (energy not in test location / energy in test location)

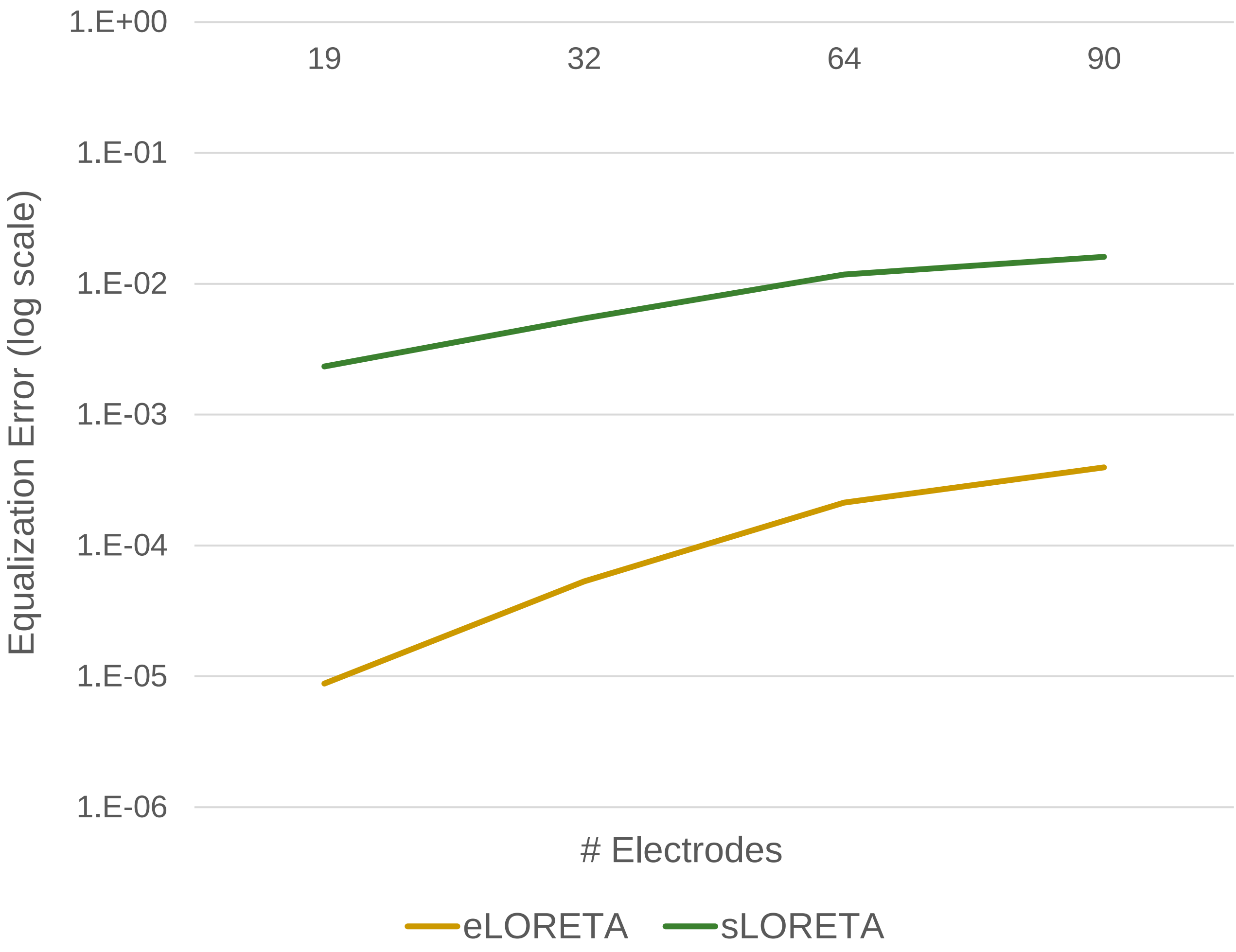
- Equalization Error:

variance of energy across test locations and orientations

Comparing sLORETA vs eLORETA



Comparing sLORETA vs eLORETA



Upper Bounds for Data Driven sLORETA

$$\boldsymbol{T}_q^T = \left(\boldsymbol{K}_q^T \boldsymbol{C} \boldsymbol{K}_q\right)^{-1/2} \boldsymbol{K}_q^T \boldsymbol{C}$$

- $\boldsymbol{C}=(\boldsymbol{K}\boldsymbol{K}^T)^+ \rightarrow$ sLORETA
  (no a-priori, maximum entropy, model-driven covariance)
- $\boldsymbol{C}=(\boldsymbol{K}_q\boldsymbol{K}_q^T)^+$ $\rightarrow$ Position
- $\boldsymbol{C}=(\boldsymbol{K}_\theta\boldsymbol{K}_\theta^T)^+$ $\rightarrow$ Orientation
- $\boldsymbol{C}=(\boldsymbol{k}_{q\theta}\boldsymbol{k}_{q\theta}^T)^+$ $\rightarrow$ Position + Orientation

Results (1/2)

Ratio: 1495

SPREAD ERROR

electrodes	eLORETA	sLORETA	Pos+Orient	Pos	Orient
19	429.0615974	483.6912568	0.32347742	0.47575874	406.1040238
32	281.2597383	350.1400515	0.19188051	0.27920945	296.7390919
64	158.5638184	219.6632052	0.08993596	0.11086595	167.5994111
90	109.2754763	151.5266824	0.02709140	0.02887675	107.3542492

Ratio: 5593

Results (2/2)

Ratio: 103

EQUALIZATION ERROR

Electrodes	eLORETA	sLORETA	Pos+Orient	Pos	Orient
19	0.00000881	0.00233484	0.00002256	0.00003151	0.00794155
32	0.00005328	0.00545325	0.00001283	0.00001824	0.01568172
64	0.00021251	0.01177894	0.00000722	0.00001068	0.03443352
90	0.00039541	0.01606805	0.00000190	0.00000194	0.04777370

Ratio: 8457