1 Flow variables and units (SI)

Flow variables and their units:

1. Density ρ is $\mathrm{kg/m^3}$
2. Pressure p is Pa (is $\mathrm{N/m^2}$ is $\mathrm{J/m^3}$)
3. Flow velocity v is m/s
4. Total energy per unit volume $E = \rho e = \rho\varepsilon_{\mathrm{int}} + \rho v^2/2$ is $\mathrm{J/m^3}$
5. Total specific energy e is J/kg
6. Total specific internal energy ε is J/kg
7. Mass of the constituent species m is kg

2 Euler equations

For a two-dimensional flow of a single-species inviscid gas the compressible Euler equations governing such a flow are given by

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}_x + \frac{\partial}{\partial y}\mathbf{f}_y = 0. \tag{1}$$

The vector of conservative variables $\mathbf{u} \in \mathbb{R}^4$ is given by

$$\mathbf{u} = (\rho, \rho v_x, \rho v_y, E)^{\mathrm{T}}, \tag{2}$$

The inviscid fluxes are given by

$$\mathbf{f}_x = \left(\rho v_x, \rho v_x^2 + p, \rho v_x v_y, (E+p)v_x\right)^{\mathrm{T}}, \tag{3}$$

$$\mathbf{f}_y = \left(\rho v_y, \rho v_x v_y, \rho v_y^2 + p, (E+p)v_y\right)^{\mathrm{T}}. \tag{4}$$

Instead of E one can also write ρe.

Re-writing via scaled variables and reference quantities, we obtain

$$\frac{\partial}{t_{ref}\partial\hat{t}}\mathbf{u} + \frac{\partial}{L_{ref}\partial\hat{x}}\mathbf{f}_x + \frac{\partial}{L_{ref}\partial\hat{y}}\mathbf{f}_y = 0, \tag{5}$$

$$\mathbf{u} = \rho_{ref}\left(\hat{\rho}, v_{ref}\hat{\rho}\hat{v}_x, v_{ref}\hat{\rho}\hat{v}_y, p_{ref}\hat{\rho}\hat{e}\right), \tag{6}$$

$$\mathbf{f}_x = \left(\rho_{ref}v_{ref}\hat{\rho}\hat{v}_x, p_{ref}\hat{\rho}\hat{v}_x^2 + p_{ref}\hat{p}, p_{ref}\hat{\rho}\hat{v}_x\hat{v}_y, v_{ref}p_{ref}(\hat{\rho}\hat{e}+\hat{p})\hat{v}_x\right), \tag{7}$$

$$\mathbf{f}_y = \left(\rho_{ref}v_{ref}\hat{\rho}\hat{v}_y, p_{ref}\hat{\rho}\hat{v}_x\hat{v}_y, p_{ref}\hat{\rho}\hat{v}_y^2 + p_{ref}\hat{p}, v_{ref}p_{ref}(\hat{\rho}\hat{e}+\hat{p})\hat{v}_y\right). \tag{8}$$

Accounting for the fact that $t_{ref} = L_{ref}/v_{ref}$, we can write:

$$\frac{\partial}{\partial\hat{t}}\hat{\mathbf{u}} + \frac{\partial}{\partial\hat{x}}\hat{\mathbf{f}}_x + \frac{\partial}{\partial\hat{y}}\hat{\mathbf{f}}_y = 0, \tag{9}$$

$$\mathbf{u} = \left(\hat{\rho}, \hat{\rho}\hat{v}_x, \hat{\rho}\hat{v}_y, \hat{\rho}\hat{e},\right), \tag{10}$$

$$\mathbf{f}_x = \left(\hat{\rho}\hat{v}_x, \hat{\rho}\hat{v}_x^2 + \hat{p}, \hat{\rho}\hat{v}_x\hat{v}_y, (\hat{\rho}\hat{e}+\hat{p})\hat{v}_x\right), \tag{11}$$

$$\mathbf{f}_y = \left(\hat{\rho}\hat{v}_y, \hat{\rho}\hat{v}_x\hat{v}_y, \hat{\rho}\hat{v}_y^2 + \hat{p}, (\hat{\rho}\hat{e}+\hat{p})\hat{v}_y\right). \tag{12}$$

So the scaled Euler equations are identical to the non-scaled ones, as all reference quantities cancel out.

2.1 Scaling of specific heats

We have (for a single-component flow) $T = T_{ref}\hat{T} = m_{ref}p_{ref}/\rho_{ref}/k\hat{T}$, $\hat{T} = \hat{p}/\rho_{ref}$ (since at $T = T_{ref}$ and $n = n_{ref}$ the flow density is ρ_{ref} and the pressure is p_{ref}). Scaling of specific heats:

$$c_v(T) = \frac{\partial \varepsilon}{\partial T} = \frac{p_{ref}}{\rho_{ref}T_{ref}}\frac{\partial \hat{\varepsilon}}{\partial \hat{T}} = c_{v,ref}\hat{c}_v. \tag{13}$$

So $c_{v,ref} = p_{ref}/(\rho_{ref}T_{ref}) = k/m_{ref}$. Mayer's relation in scaled form then reads that $\hat{c}_p = \hat{c}_v + 1$.

3 Navier–Stokes equations

Let us consider the (multi-temperature) Navier–Stokes equations in general form:

$$\frac{d}{dt}\rho_s + \rho_s\nabla\cdot\mathbf{v} + \nabla\cdot(\rho_s\mathbf{V}_s) = 0, \quad s = 1,\ldots,N_s \tag{14}$$

$$\rho\frac{d}{dt}\mathbf{v} + \nabla\cdot\mathbf{P} = 0, \tag{15}$$

$$\rho\frac{d}{dt}U + \nabla\cdot\mathbf{q} + \mathbf{P} : \nabla\mathbf{v} = 0, \tag{16}$$

$$\rho\frac{d}{dt}e_s^v + \nabla\cdot\mathbf{q}_\mathbf{s}^\mathbf{v} = e_s^v\nabla\cdot(\rho_s\mathbf{V}_s). \tag{17}$$

With scaling:

$$\frac{\rho_{ref}}{t_{ref}}\frac{d}{d\hat{t}}\hat{\rho}_s + \frac{\rho_{ref}v_{ref}}{L_{ref}}\hat{\rho}_s\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{v}} + \frac{\rho_{ref}v_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot(\rho_s\hat{\mathbf{V}}_s) = 0, \quad s = 1,\ldots,N_s \tag{18}$$

$$\frac{\rho_{ref}v_{ref}}{t_{ref}}\hat{\rho}\frac{d}{d\hat{t}}\hat{\mathbf{v}} + \frac{p_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{P}} = 0, \tag{19}$$

$$\frac{p_{ref}}{t_{ref}}\hat{\rho}\frac{d}{d\hat{t}}\hat{U} + \frac{q_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{q}} + \frac{p_{ref}v_{ref}}{L_{ref}}\hat{\mathbf{P}} : \nabla_{\hat{\mathbf{x}}}\hat{\mathbf{v}} = 0, \tag{20}$$

$$\frac{p_{ref}}{t_{ref}}\frac{d}{dt}\hat{\rho}\hat{e}_s^v + \frac{q_{ref}}{L_{ref}}\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{q}}_s^v = p_{ref}v_{ref}\hat{e}_s\nabla\cdot(\rho_s\mathbf{V}_s). \tag{21}$$

So $q_{ref} = p_{ref}v_{ref}$. Since $v_{ref} = \sqrt{\frac{p_{ref}}{\rho_{ref}}}$, we get in scaled form

$$\frac{d}{d\hat{t}}\hat{\rho}_s + \hat{\rho}_s\nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{v}} + \nabla_{\hat{\mathbf{x}}}\cdot(\rho_s\hat{\mathbf{V}}_s) = 0, \quad s = 1,\ldots,N_s \tag{22}$$

$$\hat{\rho}\frac{d}{d\hat{t}}\hat{\mathbf{v}} + \nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{P}} = 0, \tag{23}$$

$$\hat{\rho}\frac{d}{d\hat{t}}\hat{U} + \nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{q}} + \hat{\mathbf{P}} : \nabla_{\hat{\mathbf{x}}}\hat{\mathbf{v}} = 0, \tag{24}$$

$$\frac{d}{dt}\hat{\rho}\hat{e}_s^v + \nabla_{\hat{\mathbf{x}}}\cdot\hat{\mathbf{q}}_s^v = \hat{e}_s\nabla_{\hat{\mathbf{x}}}\cdot(\hat{\rho}_s\hat{\mathbf{V}}_s). \tag{25}$$

3.1 Pressure tensor scaling

For the pressure tensor we have the generic expression

$$\mathbf{P} = p\mathbf{I} - 2\mu\mathbf{S} - \zeta \cdot \nabla\mathbf{v}\mathbf{I}. \tag{26}$$

We write in scaled form

$$\hat{\mathbf{P}} = \frac{1}{p_{ref}}\mathbf{P} = \hat{\mathbf{P}} = \hat{p}\mathbf{I} - 2\hat{\mu}\hat{\mathbf{S}} - \hat{\zeta} \cdot \nabla_{\hat{\mathbf{x}}}\hat{\mathbf{v}}\mathbf{I}. \tag{27}$$

For a consistent scaling we thus need

$$\mu_{ref} = p_{ref}\frac{L_{ref}}{v_{ref}} = p_{ref}t_{ref}, \tag{28}$$

$$\zeta_{ref} = p_{ref}\frac{L_{ref}}{v_{ref}} = p_{ref}t_{ref}. \tag{29}$$

3.2 Diffusion velocity scaling

For the diffusion velocity the generic expression is

$$\mathbf{V}_s = -\sum_p D_{sp}\mathbf{d}_{sp} - D_{T,s}\nabla \ln T - \sum_p D^p_{T^v,s}\nabla \ln T^v_p. \tag{30}$$

We re-write without the logarithms:

$$\mathbf{V}_s = -\sum_p D_{sp}\mathbf{d}_{sp} - D_{T,s}\frac{\nabla T}{T} - \sum_p D^p_{T^v,s}\frac{\nabla T^v_p}{T^v_p}. \tag{31}$$

We have for the diffusive driving forces $\mathbf{d}_{sp}$:

$$\mathbf{d}_{sp} = \nabla\left(\frac{n_s}{n}\right) + \left(\frac{n_s}{n} - \frac{\rho_s}{\rho}\right)\nabla \ln p. \tag{32}$$

We re-write them in terms of the gradients of the primitive variables:

$$\mathbf{d}_{sp} = \frac{(n/m_s)\nabla\rho_s - n_s\nabla n}{n^2} + \left(\frac{n_s}{n} - \frac{\rho_s}{\rho}\right)\frac{nk\nabla T + kT\nabla n}{p}, \tag{33}$$

where

$$\nabla n = \sum_p \frac{1}{m_p}\nabla\rho_p. \tag{34}$$

We have a scaling for the diffusive driving forces: $d_{ref} = 1/L_{ref}$.

We write in scaled form

$$\hat{\mathbf{V}}_s = \frac{1}{v_{ref}}\mathbf{V}_s = -\sum_p \hat{D}_{sp}\hat{\mathbf{d}}_{sp} - \hat{D}_{T,s}\frac{\nabla_{\hat{\mathbf{x}}}\hat{T}}{\hat{T}} - \sum_p \hat{D}^p_{T^v,s}\frac{\nabla_{\hat{\mathbf{x}}}\hat{T}^v_p}{\hat{T}^v_p}. \tag{35}$$

So for a consistent scaling we need a reference diffusion coefficient

$$D_{ref} = L_{ref}v_{ref}, \tag{36}$$

and the scaled diffusive driving forces are given by

$$\hat{\mathbf{d}}_{sp} = \frac{(\hat{n}/\hat{m}_s)\nabla_{\hat{\mathbf{x}}}\hat{\rho}_s - \hat{n}_s\nabla_{\hat{\mathbf{x}}}\hat{n}}{\hat{n}^2} + \left(\frac{\hat{n}_s}{\hat{n}} - \frac{\hat{\rho}_s}{\hat{\rho}}\right)\frac{\hat{n}\nabla_{\hat{\mathbf{x}}}\hat{T} + \hat{T}\nabla_{\hat{\mathbf{x}}}\hat{n}}{\hat{p}}. \tag{37}$$

According to (3.84) of [Nagnibeda, Kustova, 2009], we also have that

$$\hat{D}^p_{T^v,s} = 0. \tag{38}$$

This means the gradient of the vibrational temperature has no impact on the diffusion velocity.

3.3 Heat flux scaling

For the heat flux the generic expression (but already using the simplifications according to (3.85) of [Nagnibeda, Kustova, 2009]) is

$$\mathbf{q} = -\left(\lambda' + \sum_s \lambda_s^{vt}\right)\nabla T - \sum_s \left(\lambda_s^{tv} + \lambda_{ss}^{vv}\right)\nabla T_s^v - p\sum_s D_{T,s}\mathbf{d}_s + \sum_s \rho_s h_s \mathbf{V}_s. \tag{39}$$

Here $h_s = e_s + \frac{kT}{m_s}$. We introduce for brevity

$$\lambda_T = \lambda' + \sum_s \lambda_s^{vt}, \tag{40}$$

$$\lambda_{T_s^v} = \lambda_s^{tv} + \lambda_s^{vv}. \tag{41}$$

We write in scaled form

$$\hat{\mathbf{q}} = \frac{1}{p_{ref}v_{ref}}\mathbf{q} = -\hat{\lambda}_T\nabla_{\hat{\mathbf{x}}}\hat{T} - \sum_s \hat{\lambda}_{T_s^v}\nabla_{\hat{\mathbf{x}}}\hat{T}_s^v - \hat{p}\sum_s \hat{D}_{T,s}\hat{\mathbf{d}}_s + \sum_s \hat{\rho}_s\hat{h}_s\hat{\mathbf{V}}_s. \tag{42}$$

From this we get that $\lambda_{ref} = \frac{p_{ref}}{T_{ref}}v_{ref}L_{ref}$.

For the vibrational energy flux:

$$\mathbf{q_s^v} = -\lambda_s^{vt}\nabla T - \lambda_{ss}^{vv}\nabla T_s^v. \tag{43}$$

So the scaling is exactly the same:

$$\hat{\mathbf{q}}_s^v = -\hat{\lambda}_s^{vt}\nabla_{\hat{\mathbf{x}}}\hat{T} - \lambda_{ss}^{vv}\nabla_{\hat{\mathbf{x}}}\hat{T}_s^v, \tag{44}$$

with $\lambda_{ref} = \frac{p_{ref}}{T_{ref}}v_{ref}L_{ref}$.

3.4 Summary

$$\mu_{ref} = p_{ref}t_{ref}, \tag{45}$$

$$\zeta_{ref} = p_{ref}t_{ref}, \tag{46}$$

$$D_{ref} = L_{ref}v_{ref}, \tag{47}$$

$$\lambda_{ref} = \frac{p_{ref}}{T_{ref}}v_{ref}L_{ref}. \tag{48}$$

Additionally,

$$q_{ref} = p_{ref}v_{ref}, \tag{49}$$

$$d_{ref} = 1/L_{ref}. \tag{50}$$

4 Production terms