Quasipotential approximation

The general form of the BSE for the scattering amplitude is in a from

$$\mathcal{M}(k_1'k_2', k_1k_2; P) = \mathcal{V}(k_1'k_2', k_1k_2; P) + \int \frac{d^4k_2''}{(2\pi)^4}\mathcal{V}(k_1'k_2', k_1''k_2''; P)G(k_1''k_2''; P)\mathcal{M}(k_1''k_2'', k_1k_2; P), \tag{1}$$

where $\mathcal{V}$ is the potential kernel and G is the propagators for two constituent particles. Here the momentum of the system $P = k_1 + k_2 = k_1' + k_2' = k_1'' + k_2''$. It can be abbreviated as

$$\mathcal{M} = \mathcal{V} + \mathcal{V}G\mathcal{M}. \tag{2}$$

The Gross form of proposed quasipotential propagators for particles 1 and 2 with mass m_1 and m_2 written down in the center of mass frame where $P = (W, \mathbf{0})$ with particle 2 being on shell are

$$g = 2\pi i\frac{\delta^+(k_2^2 - m_2^2)}{k_1^2 - m_1^2} = 2\pi i\frac{\delta(k_2^0 - E_2)}{2E_2[(W - E_2)^2 - E_1^2]}, \tag{3}$$

where $k_1 = (k_1^0, \boldsymbol{k}) = (E_1, \boldsymbol{k})$, $k_2 = (k_2^0, -\boldsymbol{k}) = (W - E_1, -\boldsymbol{k})$ with $E_1 = \sqrt{m_1^2 + |\boldsymbol{k}|^2}$.

With the define of $G_0 = g/(2\pi i)$, the four-dimensional BSE can be reduced to a three-dimensional equation in center of mass frame

$$i\mathcal{M}(\boldsymbol{k}', \boldsymbol{k}) = i\mathcal{V}(\boldsymbol{k}', \boldsymbol{k}) + \int \frac{d\boldsymbol{k}''}{(2\pi)^3}i\mathcal{V}(\boldsymbol{k}', \boldsymbol{k}'')G_0(\boldsymbol{k}'')i\mathcal{M}(\boldsymbol{k}'', \boldsymbol{k}), \tag{4}$$

Note: the $i\mathcal{M}$ and $i\mathcal{V}$ are usually real. In the center of mass frame. We choose $\boldsymbol{k}_2 = \boldsymbol{k}$ and $\boldsymbol{k}_1 = -\boldsymbol{k}$.

Partial-wave expansion

To reduce the equation to one-dimensional equation, we apply the partial wave expansion,

$$\begin{aligned}
\mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k}) &\equiv \langle\theta'\phi', \lambda'|\mathcal{V}|\theta\phi, \lambda\rangle = \sum_{J'M'JM}\langle\theta'\phi', \lambda'\rangle\langle J'M'|\mathcal{V}|JM, \lambda\rangle\langle JM, \lambda|\theta\phi, \lambda\rangle \\
&= \sum_{JM} N_J^2 D^{J*}_{M\lambda'}(\phi', \theta', 0)\mathcal{V}^{JM}_{\lambda'\lambda}(\mathrm{k}', \mathrm{k})D^J_{M\lambda}(\phi, \theta, 0) \\
\mathcal{V}^{JM}_{\lambda'\lambda}(\mathrm{k}', \mathrm{k}) &= N_J^2\int d\Omega' d\Omega D^J_{M,\lambda'}(\phi', \theta', 0)\mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k})D^{J*}_{M,\lambda}(\phi, \theta, 0) \\
\rightarrow \mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k}) &= \sum_{JM} N_J^2 D^{J*}_{M\lambda'}(0, \theta_{k',k}, 0)\mathcal{V}^{JM}_{\lambda'\lambda}(\mathrm{k}', \mathrm{k})D^J_{M\lambda}(0, 0, 0) \\
&= \sum_{JM} N_J^2 d^J_{M\lambda'}(\theta_{k',k})\mathcal{V}^{JM}_{\lambda'\lambda}(\mathrm{k}', \mathrm{k})\delta_{M\lambda} = \sum_J N_J^2 d^J_{\lambda,\lambda'}(\theta_{k',k})\mathcal{V}^J_{\lambda'\lambda}(\mathrm{k}', \mathrm{k}) \\
\mathcal{V}^J_{\lambda'\lambda}(\mathrm{k}', \mathrm{k}) &= 2\pi\int d\cos\theta_{k,k'} d^J_{\lambda,\lambda'}(\theta_{k',k})\mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k})
\end{aligned} \tag{5}$$

where $N_J = \sqrt{\frac{2J+1}{4\pi}}$, $\int d\Omega D^{J*}_{\lambda_1,\lambda_2}(\phi, \theta, 0)D^{J'*}_{\lambda_1',\lambda_2'}(\phi, \theta, 0) = \frac{4\pi}{2J+1}$, and $\int_{-1}^{1} d^{J'}_{\lambda,\lambda'}(\theta')d^J_{\lambda,\lambda'}(\theta') = \frac{2}{2J+1}\delta_{JJ'}$ are used. The momenta are chosen as $k_2 = (E_2, 0, 0, \mathrm{k})$, $k_1 = (W - E_2, 0, 0, -\mathrm{k})$ and $k_2' = (E_2', \mathrm{k}'\sin\theta_{k,k'}, 0, \mathrm{k}'\cos\theta_{k,k'})$, $k = (W - E_2, -\mathrm{k}'\sin\theta_{k,k'}, 0, -\mathrm{k}'\cos\theta_{k,k'})$ with $\mathrm{k} = |\boldsymbol{k}|$ and $\mathrm{k}' = |\boldsymbol{k}'|$.

NOTE: Which particle is chosen to parallel to z axis is related to the order of λ and λ' in $d^J_{\lambda'\lambda}(\theta_{k,k'})$, so it can not be chosen arbitrarily. And the definition of helicity is also dependent of the definition of $\boldsymbol{k}_{1,2}$. Here, $\lambda = \lambda_2 - \lambda_1$ and $\lambda_1 = -s_1$, $\lambda_2 = s_2$. The scattering amplitudes $\mathcal{M}$ has analogous relations.

Now we have the partial wave BS equation,

$$
\begin{aligned}
i\mathcal{M}^J_{\lambda',\lambda}(\mathrm{k}',\mathrm{k}) &= 2\pi\int d\cos\theta' d^J_{\lambda,\lambda'}(\theta')\left[i\mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k})+\int\frac{d\boldsymbol{k}''}{(2\pi)^3}i\mathcal{V}_{\lambda'\lambda''}(\boldsymbol{k}',\boldsymbol{k}'')G_0(\boldsymbol{k}'')i\mathcal{M}_{\lambda''\lambda}(\boldsymbol{k}'',\boldsymbol{k})\right]\\
&= i\mathcal{V}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k})+2\pi\int d\cos\theta' d^J_{\lambda,\lambda'}(\theta')\int\frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}\\
&\cdot\int d\Omega''\sum_{JMJ'M'}N_J^4 D^{J'*}_{M'\lambda'}(0,\theta',0)i\mathcal{V}^{J'}_{\lambda'\lambda''}(\mathrm{k}',\mathrm{k}'')D^{J'}_{M'\lambda''}(\phi'',\theta'',0)G_0(\mathrm{k}'')D^{J*}_{M\lambda''}(\phi'',\theta'',0)i\mathcal{M}_{\lambda''\lambda}(\mathrm{k}'',\mathrm{k})D^J_{M\lambda}(0,0,0)\\
&= i\mathcal{V}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k})+2\pi\int d\cos\theta' d^J_{\lambda,\lambda'}(\theta')\int\frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}\\
&\cdot\int d\cos\theta'' d\phi\sum_{JMJ'M'}N_J^4 d^{J'}_{M'\lambda'}(\theta')i\mathcal{V}^{J'}_{\lambda'\lambda''}(\mathrm{k}',\mathrm{k}'')e^{-iM'\phi''}d^{J'}_{M'\lambda''}(\theta'')G_0(\mathrm{k}'')e^{iM\phi''}d^J_{M\lambda''}(\theta'')i\mathcal{M}_{\lambda''\lambda}(\mathrm{k}'',\mathrm{k})\delta_{M\lambda}\\
&= i\mathcal{V}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k})+\int d\cos\theta' d^J_{\lambda,\lambda'}(\theta')\int\frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}\\
&\cdot\int d\phi\sum_{JM}N_J^2 d^J_{M\lambda'}(\theta')i\mathcal{V}^J_{\lambda'\lambda''}(\mathrm{k}',\mathrm{k}'')e^{-iM\phi''}G_0(\mathrm{k}'')e^{iM\phi''}i\mathcal{M}_{\lambda''\lambda}(\mathrm{k}'',\mathrm{k})\delta_{M\lambda}\\
&= i\mathcal{V}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k})+\int d\cos\theta' d^J_{\lambda,\lambda'}(\theta')\int\frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}2\pi\sum_J N_J^2 d^J_{\lambda\lambda'}(\theta')i\mathcal{V}^J_{\lambda'\lambda''}(\mathrm{k}',\mathrm{k}'')G_0(\mathrm{k}'')i\mathcal{M}_{\lambda''\lambda}(\mathrm{k}'',\mathrm{k})\\
&= i\mathcal{V}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k})+\sum_J\int\frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}i\mathcal{V}^J_{\lambda'\lambda''}(\mathrm{k}',\mathrm{k}'')G_0(\mathrm{k}'')i\mathcal{M}_{\lambda''\lambda}(\mathrm{k}'',\mathrm{k})
\end{aligned}
$$

Fixed parity

For a helicity state $|J,\lambda\rangle=|J,\lambda_1\lambda_2\rangle$ fulfill the party property,

$$
P|J,\lambda\rangle=P|J,\lambda_1\lambda_2\rangle=\eta_1\eta_2(-1)^{J-s_1-s_2}|J,-\lambda_1-\lambda_2\rangle\equiv\tilde{\eta}|J-\lambda\rangle \tag{6}
$$

The construction of normalized states with parity $\pm$ is now straightforward:

$$
\begin{aligned}
|J,\lambda;\pm\rangle &= \frac{1}{\sqrt{2}}(|J,+\lambda\rangle\pm\tilde{\eta}|J,-\lambda\rangle\\
\Rightarrow P|J,\lambda;\pm\rangle &= \frac{1}{\sqrt{2}}(\tilde{\eta}|J,-\lambda\rangle\pm|J,+\lambda\rangle)=\pm\frac{1}{\sqrt{2}}(\pm\tilde{\eta}|J,-\lambda\rangle+|J,\lambda\rangle)=\pm|J,\lambda;\pm\rangle
\end{aligned} \tag{7}
$$

$$
\mathcal{M}^{J\pm}_{\lambda'\lambda}=\langle J,\lambda';\pm|\mathcal{M}|J,\lambda;\pm\rangle=\mathcal{M}^J_{\lambda'\lambda}\pm\tilde{\eta}\mathcal{M}^J_{\lambda'-\lambda},\quad \mathcal{M}^{J\pm}_{\lambda'-\lambda}=\pm\tilde{\eta}\mathcal{M}^{J\pm}_{\lambda'\lambda}\equiv\eta\mathcal{M}^{J\pm}_{\lambda'\lambda},\ \ \mathcal{M}^{J\pm}_{-\lambda'\lambda}=\pm\tilde{\eta}'\mathcal{M}^{J\pm}_{\lambda'\lambda}\equiv\eta'\mathcal{M}^{J\pm}_{\lambda'\lambda} \tag{8}
$$

with $\eta=PP_1P_2(-1)^{J_1+J_2-J}=P(-1)^{1/2+J}$.
The potential $\mathcal{V}^{J^P}_{\lambda'\lambda}$ has analogous relations.

$$
\begin{aligned}
& i\mathcal{M}_{\lambda\lambda'}=i\mathcal{V}_{\lambda\lambda'}+\sum_{\lambda''}i\mathcal{V}_{\lambda\lambda''}G_0 i\mathcal{M}_{\lambda''\lambda'},\quad \eta' i\mathcal{M}_{\lambda-\lambda'}=\eta' i\mathcal{V}_{\lambda-\lambda'}+\sum_{\lambda''}i\mathcal{V}_{\lambda\lambda''}G_0\eta' i\mathcal{M}_{\lambda''-\lambda'}\\
& \Rightarrow i\mathcal{M}^{J^P}_{\lambda\lambda'}=i\mathcal{V}^{J^P}_{\lambda\lambda'}+\sum_{\lambda''}i\mathcal{V}_{\lambda\lambda''}G_0 i\mathcal{M}^{J^P}_{\lambda''\lambda'},\quad i\mathcal{M}^{J^P}_{\lambda\lambda'}=i\mathcal{V}^{J^P}_{\lambda\lambda'}+\sum_{\lambda''}i\mathcal{V}_{\lambda-\lambda''}G_0 i\mathcal{M}^{J^P}_{-\lambda''\lambda'}=iV^{J^P}_{\lambda\lambda'}+\sum_{\lambda''}i\mathcal{V}_{\lambda-\lambda''}G_0\eta'' i\mathcal{M}^{J^P}_{-\lambda''\lambda'}\\
& \Rightarrow i\mathcal{M}^{J^P}_{\lambda\lambda'}=i\mathcal{V}^{J^P}_{\lambda\lambda'}+\frac{1}{2}\sum_{\lambda''}i\mathcal{V}^{J^P}_{\lambda\lambda''}G_0 i\mathcal{M}^{J^P}_{\lambda''\lambda'},
\end{aligned} \tag{9}
$$

As shown in Eq.~(9), the amplitudes are not independent. If we only keep the independent amplitudes, the equation for definite parity can be written as

$$
\begin{aligned}
i\mathcal{M}^{J^P}_{ij} &= i\mathcal{V}^{J^P}_{ij}+\frac{1}{2}\sum_{\lambda''}i\mathcal{V}^{J^P}_{i\lambda''}G_0 i\mathcal{M}^{J^P}_{\lambda''j}=i\mathcal{V}^{J^P}_{ij}+\frac{1}{2}i\mathcal{V}^{J^P}_{i0}Gi\mathcal{M}^{J^P}_{0j}+\sum_{k\neq 0}i\mathcal{V}^{J^P}_{ik}G_0 i\mathcal{M}^{J^P}_{kj},\\
\Rightarrow i\mathcal{M}^{J^P}_{ij} &= i\mathcal{V}^{J^P}_{ij}+\sum_k i\mathcal{V}^{J^P}_{ik}G_0 i\mathcal{M}^{J^P}_{kj}.
\end{aligned} \tag{10}
$$

where i, j, k are the indices for the independent amplitudes. And we redefine

$$f_i f_j \mathcal{M}^{J^P}_{\lambda'\lambda} \to \mathcal{M}^{J^P}_{ij}$$

with $f_0 = \frac{1}{\sqrt{2}}$ and $f_{i\neq 0} = 1$ with 0 for the amplitudes with $\lambda_1 = \lambda_2 = 0$.

The Bethe-Saltpeter equation for partial-wave amplitude with fixed spin-parity J^P reads ,

$$i\mathcal{M}^{J^P}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k}) = i\mathcal{V}^{J^P}_{\lambda',\lambda}(\mathrm{k}',\mathrm{k}) + \sum_{\lambda''}\int \frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}\, i\mathcal{V}^{J^P}_{\lambda'\lambda''}(\mathrm{k}',\mathrm{k}'')G_0(\mathrm{k}'')i\mathcal{M}^{J^P}_{\lambda''\lambda}(\mathrm{k}'',\mathrm{k}). \tag{11}$$

Note that here the sum extends only over ndices for the independent amplitudes. The partial wave potential is defined as

$$i\mathcal{V}^{J^P}_{\lambda'\lambda}(\mathrm{k}',\mathrm{k}) = 2\pi\int d\cos\theta\,[d^J_{\lambda\lambda'}(\theta)i\mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k}) + \eta d^J_{-\lambda\lambda'}(\theta)i\mathcal{V}_{\lambda'-\lambda}(\boldsymbol{k}',\boldsymbol{k})], \tag{12}$$

Or, with the independent amplitudes as

$$i\mathcal{M}^{J^P}_{ij}(\mathrm{k}',\mathrm{k}) = i\mathcal{V}^{J^P}_{ij}(\mathrm{k}',\mathrm{k}) + \sum_{k}\int \frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}\, i\mathcal{V}^{J^P}_{ik}(\mathrm{k}',\mathrm{k}'')G_0(\mathrm{k}'')i\mathcal{M}^{J^P}_{kj}(\mathrm{k}'',\mathrm{k}). \tag{13}$$

$$i\mathcal{V}^{J^P}_{ij}(\mathrm{k}',\mathrm{k}'') = f_i f_j 2\pi\int d\cos\theta\,[d^J_{\lambda\lambda'}(\theta)i\mathcal{V}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k}) + \eta d^J_{-\lambda\lambda'}(\theta)i\mathcal{V}_{\lambda'-\lambda}(\boldsymbol{k}',\boldsymbol{k})]. \tag{14}$$

Note here the $f_i f_j$is also incorporated. Additionally, the form factors for the interacting particles are also included in the potential, modifying it as $\mathcal{V} \to f(k')\mathcal{V}f(k)$. Consequently, the resulting amplitude $\mathcal{M}$ also includes these form factors.

Transformation to a matrix equation

Now We have a integral equation with singularity in $G_0 = \frac{1}{2E((s-E)^2-\omega^2)}$ at $W = E_1 + E_2$. This singularity can be isolated as,

$$\begin{aligned}
i\mathcal{M}^{J^P}(\mathrm{k}',\mathrm{k}) &= i\mathcal{V}^{J^P}(\mathrm{k},\mathrm{k}') + \int \frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3} i\mathcal{V}^{J^P}(\mathrm{k},\mathrm{k}'')G_0(\mathrm{k}'')i\mathcal{M}^{J^P}(\mathrm{k}'',\mathrm{k}') \\
&= i\mathcal{V}^{J^P} + \mathcal{P}\int \frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3} i\mathcal{V}^{J^P}G_0\mathcal{M}^{J^P} - \pi i\int \frac{\mathrm{k}''^2 d\mathrm{k}''}{(2\pi)^3}\mathcal{V}^{J^P}_o\delta G_0\mathcal{M}^{J^P}_o \\
&= i\mathcal{V}^{J^P} + \int \frac{d\mathrm{k}''}{(2\pi)^3}\left[\mathrm{k}''^2 G_0 i\mathcal{V}^{J^P}\mathcal{M}^{J^P} - \frac{M\mathcal{V}^{J^P}_o\mathcal{M}^{J^P}_o}{\mathrm{k}''^2 - \bar{\mathrm{k}''}^2}\right] - i\frac{\bar{\mathrm{k}''}^2\delta\bar{G}_0}{8\pi^2}\mathcal{V}^{J^P}_o\mathcal{M}^{J^P}_o \\
&= i\mathcal{V}^{J^P} + \int \frac{d\mathrm{k}''}{(2\pi)^3}\mathrm{k}''^2 G_0 i\mathcal{V}^{J^P}\mathcal{M}^{J^P} - [\int \frac{d\mathrm{k}''}{(2\pi)^3}\frac{M}{\mathrm{k}''^2 - \bar{\mathrm{k}''}^2} + i\frac{\bar{\mathrm{k}''}^2\delta\bar{G}_0}{8\pi^2}]\mathcal{V}^{J^P}_o\mathcal{M}^{J^P}_o
\end{aligned}$$

with $\delta G_0 = \frac{\delta(s-E-\omega))}{2E(s-E+\omega)}\theta(s-m_1-m_2) = \delta\bar{G}_0\delta(\mathrm{k}''-\bar{\mathrm{k}''}) = \frac{1}{4W\bar{\mathrm{k}''}}\delta(\mathrm{k}''-\bar{\mathrm{k}''})\theta(s-m_1-m_2)$, $M = [\mathrm{k}''^2(\mathrm{k}''^2 - \bar{\mathrm{k}''}^2)G_0]_{\mathrm{k}''\to\bar{\mathrm{k}''}}\theta(s-m_1-m_2) = -\frac{\bar{\mathrm{k}''}^2}{2W}\theta(s-m_1-m_2)$.

We have

$$Im\, G = -\rho/2 = -\frac{\bar{\mathrm{k}''}}{32\pi^2 W}. \tag{15}$$

It suggests the unitary is satisfied if the potential $i\mathcal{V}$ is real.

$$-T^\dagger\rho T = 2T^\dagger\, ImG\, T = 2T^\dagger(-ImT^{-1})T = 2T^\dagger\frac{1}{2i}(T^{\dagger-1} - T^{-1})T = i(T - T^\dagger) \tag{16}$$

where $T = i\mathcal{M}$.

With the Gauss discretization, the one-dimensional equation can be transformed as a matrix equation as

$$iM_{ik}^{J^P} = iV_{ik}^{J^P} + \sum_{j=0}^{N} iV_{ij}^{J^P} G_j iM_{jk}^{J^P} \Rightarrow M^{J^P} = V^{J^P} + V^{J^P} G M^{J^P} \tag{17}$$

$$G_j = \begin{cases} -\frac{i\bar{q}}{32\pi^2 W} + \sum_j \left[\frac{w(q_j)}{(2\pi)^3} \frac{\bar{q}^2}{2W(q_j^2-\bar{q}^2)}\right] & \text{for } j=0, \text{ if } Re(W) > m_1+m_2, \\ \frac{w(q_j)}{(2\pi)^3} \frac{q_j^2}{2E(q_j)[(W-E(q_j))^2-\omega^2(q_j)]} & \text{for } j \neq 0 \end{cases}$$

where $\bar{q} = \frac{1}{2W}\sqrt{[W^2-(m_1+m_2)^2][W^2-(m_1-m_2)^2]}$. The indices i, j, k is for discrete momentum values, independent helicities, and coupled channels.

The default dimension is $[G]=1$. Recalling that a factor of $2m$ should be included if a constituent particle is a fermion, we have $[G] = \mathrm{GeV}^{n_f} \to [V] = [M] = \mathrm{GeV}^{-n_f}$, with n_f being the number of fermions. Therefore, under our convention where $\bar{u}u = 1$, the dimension of the potential must satisfy the above requirements. This criterion can be employed to verify the consistency of the Lagrangian and the derived potential.

Hence, for the channels above its thresholds, the matrix have an extra dimension.
We take two channel as example to explain the coupled-channel equation. The region of W is divided as $W < m_1, m_1 < W < m_2$ and $W > m_2$.

$$V = \begin{pmatrix} V_{11}^{NN} & V_{12}^{NN} \\ V_{21}^{NN} & V_{22}^{NN} \end{pmatrix}, \quad G = \begin{pmatrix} G_1^N & 0 \\ 0 & G_2^N \end{pmatrix}, \quad W < m_1, \tag{18}$$

$$V = \begin{pmatrix} V_{11}^{N+1N+1} & V_{12}^{N+1N} \\ V_{21}^{NN+1} & V_{22}^{NN} \end{pmatrix}, \quad G = \begin{pmatrix} G_1^{N+1} & 0 \\ 0 & G_2^N \end{pmatrix}, \quad m_1 < W < m_2 \tag{19}$$

$$V = \begin{pmatrix} V_{11}^{N+1N+1} & V_{12}^{N+1N+1} \\ V_{21}^{N+1N+1} & V_{22}^{N+1N+1} \end{pmatrix}, \quad G = \begin{pmatrix} G_1^{N+1} & 0 \\ 0 & G_2^{N+1} \end{pmatrix}, \quad W > m_2 \tag{20}$$

Code

Attention: The following details are specific to the old version of the code, which includes Fortran code and Julia code versions prior to v0.2.4. In the new version, the treatment described below is obsolete.

In old code, we choose $\hat{V}^{J^P} = f(k')V^{J^P}f(k)/4\pi$, $\hat{G} = 4\pi G$, and $\hat{M}^{J^P} = f(k')M^{J^P}f(k)/4\pi$. The form factors are also included in to the ampltudes and the potential kernel. Hence, the qBSE becomes

$$\hat{M}^{J^P} = \hat{V}^{J^P} + \hat{V}^{J^P} G \hat{M}^{J^P}. \tag{21}$$

Such convention is consistent with that in the chiral unitary approach.

$$\begin{aligned} \hat{V}^{J^P} &= V^{J^P}/4\pi = iV_{\lambda'\lambda''}^{J^P}(\mathrm{p}', \mathrm{p}'')/4\pi = f_i f_j iV_{\lambda'\lambda}^{J^P}(\mathrm{p}', \mathrm{p})/4\pi \\ &= \frac{1}{2} f_i f_j \int d\cos\theta \, [d_{\lambda\lambda'}^J(\theta) iV_{\lambda'\lambda}(\boldsymbol{p}', \boldsymbol{p}) + \eta d_{-\lambda\lambda'}^J(\theta) iV_{\lambda'-\lambda}(\boldsymbol{p}', \boldsymbol{p})], \end{aligned} \tag{22}$$

$$\hat{G}_j = \begin{cases} -\frac{i\bar{q}}{8\pi W} + \sum_j \left[\frac{w(q_j)}{2\pi^2} \frac{\bar{q}^2}{2W(q_j^2-\bar{q}^2)}\right] & \text{for } j=0, \text{ if } Re(W) > m_1+m_2, \\ \frac{w(q_j)}{2\pi^2} \frac{q_j^2}{2E(q_j)[(W-E(q_j))^2-\omega^2(q_j)]} & \text{for } j \neq 0 \end{cases}$$

Physical observable

After extend the energy in the center of mass frame W into complex energy plane as z, the pole can be found by variation of z to satisfy

$$|1 - V(z)G(z)| = 0 \tag{23}$$

with $z = E_R + i\Gamma_R/2$.

With the obtained amplitude M^{J^P}, we can also calculate the physical observable. Note that all physical observable are at real axis, we choose the onshell momentum as

$$M_{ij}(z) = \{[(1-VG)^{-1}]V\}_{ij} \tag{24}$$

with i and k chosen as the onshell momentum, that is, 0 dimension for G, and extra dimension for V.

The cross section for the channel considered

For the open channel, the cross section can be obtained as

$$\frac{d\sigma}{d\Omega} = \frac{1}{\tilde{j}_1\tilde{j}_2}\frac{1}{64\pi^2 s}\frac{\mathrm{k}'}{\mathrm{k}}\sum_{\lambda\lambda'}|i\mathcal{M}_{\lambda\lambda'}(\boldsymbol{k}',\boldsymbol{k})|^2. \tag{25}$$

where j_1 and j_2 is the spin of the intitial particles, and we define $\tilde{j} = 2j+1$.

The total cross section can be written as

$$\sigma = \frac{1}{\tilde{j}_1\tilde{j}_2}\frac{1}{64\pi^2 s}\frac{\mathrm{k}'}{\mathrm{k}}\int d\Omega\sum_{\lambda}|i\mathcal{M}_{\lambda\lambda'}(\boldsymbol{k}',\boldsymbol{k})|^2 = \frac{1}{\tilde{j}_1\tilde{j}_2}\frac{1}{64\pi^2 s}\frac{\mathrm{k}'}{\mathrm{k}}\sum_{J,\lambda}\frac{\tilde{J}}{4\pi}|i\mathcal{M}^J_{\lambda\lambda'}(\mathrm{k}',\mathrm{k})|^2 = \frac{1}{256\pi^3 s}\frac{|\boldsymbol{k}'|}{|\boldsymbol{k}|}\sum_{J^P,ij}\frac{\tilde{J}}{\tilde{j}_1\tilde{j}_2}\left|M^{J^P}_{ij}\right|^2 \tag{26}$$

Here, k' and k are onshell momenta, so we only choose ij for the onshell momenta. In the last step the M^{J^P} is replaced by the matrix obtained in code with relation $\hat{M}^{J^P} = f(k')M^{J^P}f(k)/4\pi$. The form factors vanish due to onshelness for initial and final states of a scattering.
NOTE: $4MM'$ should be multiplied if convention $\bar{u}u=1$ adopted.

$$\begin{aligned}
\sigma \propto \sum_{J,\lambda'\lambda}|M^J_{\lambda'\lambda}|^2 &= \sum_{J,\lambda' j=0}|M^J_{\lambda' j}|^2 + \sum_{J,\lambda' j>0}\left[|M^J_{\lambda' j}|^2 + |M^J_{\lambda' -j}|^2\right] \\
&= \sum_{J^P,\lambda' j=0}\delta_{\eta,1}|\frac{1}{2}M^{J^P}_{\lambda' j}|^2 + \sum_{J,\lambda' j>0}\left[\frac{1}{4}|M^{J^+}_{\lambda' j} + M^{J^-}_{\lambda' j}|^2 + \frac{1}{4}|M^{J^+}_{\lambda' j} - M^{J^-}_{\lambda' j}|^2\right] \\
&= \sum_{J^P,i=0j=0}\delta_{\eta,1}\delta_{\eta',1}|\frac{1}{2}M^{J^P}_{ij}|^2 + \sum_{J^P,i>0j=0}2\delta_{\eta,1}|\frac{1}{2}M^{J^P}_{ij}|^2 + \sum_{J^P,\lambda' j>0}\frac{1}{2}|M^{J^P}_{\lambda' j}|^2 \\
&= \sum_{J^P,i=0j=0}\delta_{\eta,1}\delta_{\eta',1}|\frac{1}{2}M^{J^P}_{ij}|^2 + \sum_{J^P,i>0j=0}\delta_{\eta,1}|\frac{1}{\sqrt{2}}M^{J^P}_{ij}|^2 + \sum_{J^P,i=0j>0}\delta_{\eta',1}|\frac{1}{\sqrt{2}}M^{J^P}_{ij}|^2 + \sum_{J^P,i>0j>0}|M^{J^P}_{\lambda' j}|^2 \\
&= \sum_{J^P,i=0j=0}\delta_{\eta,1}\delta_{\eta',1}|\frac{1}{2}M^{J^P}_{ij}|^2 + \sum_{J^P,i>0j=0}\delta_{\eta,1}|\frac{1}{\sqrt{2}}M^{J^P}_{ij}|^2 + \sum_{J^P,i=0j>0}\delta_{\eta',1}|\frac{1}{\sqrt{2}}M^{J^P}_{ij}|^2 + \sum_{J^P,i>0j>0}|M^{J^P}_{ij}|^2 \\
&= \sum_{J^P,i\geq 0j\geq 0}|f_i f_j M^{J^P}_{ij}|^2 = \sum_{J^P,i\geq 0j\geq 0}|M^{J^P}_{ij}|^2
\end{aligned} \tag{27}$$

Argand plot

The amplitudes can be written as

$$i\mathcal{M}(\boldsymbol{k}) = -8\pi\sqrt{s}f(\boldsymbol{k}) = -\frac{8\pi\sqrt{s}}{|\boldsymbol{k}|}\sum_{JM}\frac{2J+1}{4\pi}D^{J*}_{\lambda_R,\lambda}(\phi,\theta,-\phi)a^J_{\lambda\lambda'}(|\boldsymbol{k}|)D^J_{M,\lambda'}(\phi',\theta',-\phi'). \tag{28}$$

where $a^J = -\frac{|\boldsymbol{k}|}{8\pi\sqrt{s}}\mathcal{M}(|\boldsymbol{k}|)$, which can be displayed as a trajectory in an Argand plot.

Three body decay

kinematics

Lorentz boost

Here, we consider an process $Y \to m_1 X \to m_1 m_2 m_3$.
To study a $1 \to 3$ decay with the qBSE, we need consider the center of mass frame (CMS) of Y (which is also the laboratory frame in this issue) and the $m_1 m_2$ where the qBSE is applied. The momenta of initial and final particles in the CMS of Y, remarked as lab, are

$$P^{lab} = (W, 0, 0, 0);\;\; p_1^{lab} = (E_1^{lab}, \boldsymbol{p}_1^{lab});\;\; p_2^{lab} = (E_2^{lab}, \boldsymbol{p}_2^{lab});\;\; p_3^{lab} = (E_3^{lab}, \boldsymbol{p}_3^{lab}) \tag{29}$$

The Lorentz boost from $(m, \boldsymbol{0})$ to $(E, \boldsymbol{k})$,

$$\Lambda^\mu_\nu = \frac{1}{m}\begin{pmatrix} E(\boldsymbol{k}) & k_x & k_y & k_z \\ k_x & m + \frac{k_x k_x}{E+m} & \frac{k_x k_y}{E+m} & \frac{k_x k_z}{E+m} \\ k_y & \frac{k_y k_x}{E+m} & m + \frac{k_y k_y}{E+m} & \frac{k_y k_z}{E+m} \\ k_z & \frac{k_z k_x}{E+m} & \frac{k_z k_y}{E+m} & m + \frac{k_z k_z}{E+m} \end{pmatrix}.$$

With the Lorentz boost the momenta for particle 23 in the laboratory frame $(E_{23}^{lab}, -\boldsymbol{p}_1^{lab})$ can be written with the momenta in the CMS of particles 23 $(M_{23}, \boldsymbol{0})$ as $p^{lab} = \Lambda(E_{23}^{lab}, -\boldsymbol{p}_1^{lab})p^{cm}$,

$$\begin{aligned} \boldsymbol{p}^{lab} &= \boldsymbol{p}^{cm} - \frac{\boldsymbol{p}_1^{lab}}{M_{23}}\left[\frac{-\boldsymbol{p}_1^{lab}\cdot\boldsymbol{p}^{cm}}{W - E_1^{lab}(\mathrm{p}_1^{lab}) + M_{23}} + p^{0cm}\right], \\ p^{0lab} &= \frac{1}{M_{23}}\left[(W - E_1^{lab}(\mathrm{p}_1^{lab}))p^{0cm} - \boldsymbol{p}_1^{lab}\cdot\boldsymbol{p}^{cm}\right], \end{aligned} \tag{30}$$

where the $p_{23} + p_1 = P$ is applied, and $M_{23} = \sqrt{(p_2^{lab} + p_3^{lab})^2} = \sqrt{(p_2^{cm} + p_3^{cm})^2} = \sqrt{(P - p_1)^2}$.

The momenta in CMS of 23 can also be written with the momentum in laboratory frame as $p^{cm} = \Lambda(E_{23}^{lab}, \boldsymbol{p}_1^{lab})p^{lab}$

$$\begin{aligned} \boldsymbol{p}^{cm} &= \boldsymbol{p}^{lab} + \frac{\boldsymbol{p}_1^{lab}}{M_{23}}\left[\frac{\boldsymbol{p}_1^{lab}\cdot\boldsymbol{p}^{lab}}{M - E_1^{lab}(\mathrm{p}_1^{lab}) + M_{23}} + p^{0lab}\right], \\ p^{0cm} &= \frac{1}{M_{23}}\left[(M - E_1^{lab}(\mathrm{p}_1^{lab}))p^{0lab} + \boldsymbol{p}_1^{lab}\cdot\boldsymbol{p}^{lab}\right]. \end{aligned} \tag{31}$$

For the outgoing particles 2 and 3, the $\boldsymbol{p}^{cm}$ should be set along the z axis, hence, an additional rotation is adopted as $p = R(-\theta^{cm}, -\phi^{cm})p^{cm}$

$$R(-\theta, -\phi) = \frac{1}{m}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}.$$

As described above, we perform both a Lorentz boost and a rotation. Through the Lorentz boost, the momentum p^{lab} in the lab frame transforms into p^{cm} in the center-of-mass system (CMS) of particles 2 and 3. The value of p_1^{cm} depends on p_1^{lab} and M_{23}, and the lab-frame energy E_1^{lab} can also be derived from M_{23}. Thus, p_1^{cm} is ultimately dependent only on Ω_1^{lab} and M_{23}. The momenta $p_{2,3}^{cm}$ depend on $p_{2,3}^{lab}$ and Ω_1^{lab} and can be expressed in terms of M_{23} and the spherical angle Ω_3^{cm}.

After rotation, p^{cm} becomes p in the new CMS frame of particles 2 and 3. The momenta $p_{2,3}$ of the final particles are aligned along the z-axis and depend solely on M_{23}. However, for intermediate particles, they are functions of p_3' and Ω_3'. Finally, p_1 depends on Ω_1^{lab}, M_{23}, and Ω_3^{cm}.

Amplitude

Because the $|\mathcal{M}|^2$ is invariant in different reference frame, the amplitude for the direct decay can be written with the momenta in cm frame of partilces 1 and 2 obtained with Lorentz boost and rotation, as

$$\begin{aligned}
i\mathcal{M}^d_{\lambda_1;\lambda_2,\lambda_3;\lambda}(p_1,p_2,p_3) &= i\mathcal{A}_{\lambda_1;\lambda_2,\lambda_3;\lambda}(\Lambda R p_1,p_2,p_3) = i\mathcal{A}_{\lambda_1;\lambda_2,\lambda_3;\lambda}(\Omega_1^{lab},M_{23},\Omega_3^{cm}),\quad \text{for onshell}\\
i\mathcal{M}^d_{\lambda_1;\lambda_2,\lambda_3;\lambda}(p_1,p_2',p_3') &= i\mathcal{A}_{\lambda_1;\lambda_2,\lambda_3;\lambda}(\Omega_1^{lab},\Omega_3',\mathrm{p}_3',M_{23},\Omega_3^{cm})\\
&= \sum_{JM} N_J D^{J*}_{M\lambda_{32}}(\Omega_3') i\mathcal{A}^{JM}_{\lambda_1;\lambda_2,\lambda_3;\lambda}(\Omega_1^{lab},\mathrm{p}_3',M_{23},\Omega_3^{cm}),\quad \text{for offshell}\\
i\mathcal{A}^{JM}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(\Omega_1^{lab},\mathrm{p}_2',M_{23},\Omega_3^{cm}) &= N_J\int d\Omega_3' D^J_{M,\lambda_{32}'}(\Omega_3') i\mathcal{A}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(\Omega_1^{lab},\Omega_3',\mathrm{p}_3',M_{23},\Omega_3^{cm})\\
i\mathcal{M}^Z_{\lambda_1;\lambda_2,\lambda_3;\lambda}(p_1,p_2,p_3) &= i\int\frac{d^4p_3'}{(2\pi)^4}\mathcal{T}_{\lambda_2,\lambda_3}(p_2,p_3;p_2',p_3')G(p_3')\mathcal{A}_{\lambda_1;\lambda}(p_1,p_2',p_3')\\
&= \sum_{\lambda_2'\lambda_3'}\int\frac{d^3p_3'}{(2\pi)^3} i\mathcal{T}_{\lambda_2,\lambda_3;\lambda_2',\lambda_3'}(p_2,p_3;p_2',p_3')G_0(p_3') i\mathcal{A}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(p_1,p_2',p_3')\\
&= \sum_{\lambda_2'\lambda_3'}\int\frac{d^3p_3'}{(2\pi)^3} i\mathcal{T}_{\lambda_2,\lambda_3;\lambda_2',\lambda_3'}(\Omega_3',\mathrm{p}_3',M_{23})G_0(\mathrm{p}_3') i\mathcal{A}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(\Omega_1^{lab},\Omega_3',\mathrm{p}_3',M_{23},\Omega_3^{cm})\\
&= \sum_{\lambda_2'\lambda_3'}\int\frac{\mathrm{p}_3'^2 d\mathrm{p}_3' d\Omega_3'}{(2\pi)^3}\sum_{JM} N_J^2\delta_{M\lambda_{32}} i\mathcal{T}^{JM}_{\lambda_2,\lambda_3;\lambda_2',\lambda_3'}(\mathrm{p}_3',M_{23}) D^J_{M\lambda_{32}'}(\Omega_3')\\
&\cdot\ G_0(\mathrm{p}_3')\sum_{J'M'} N_{J'} D^{J'*}_{M'\lambda_{32}'}(\Omega_3') i\mathcal{A}^{J'M'}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(\Omega_1^{lab},\mathrm{p}_3',M_{23},\Omega_3^{cm})\\
&= \sum_{\lambda_2'\lambda_3'}\int\frac{\mathrm{p}_3'^2 d\mathrm{p}_3' d\Omega_3'}{(2\pi)^3}\sum_J N_J i\mathcal{T}^J_{\lambda_2,\lambda_3;\lambda_2',\lambda_3'}(\mathrm{p}_3',M_{23})G_0(\mathrm{p}_3') i\mathcal{A}^{J\lambda_{32}}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(\Omega_1^{lab},\mathrm{p}_3',M_{23},\Omega_3^{cm})\\
&= \sum_J N_J\sum_{\lambda_2'\lambda_3'}\int\frac{\mathrm{p}_3'^2 d\mathrm{p}_3'}{(2\pi)^3} i\mathcal{T}^J_{\lambda_2,\lambda_3;\lambda_2',\lambda_3'}(\mathrm{p}_3',M_{23})G_0(\mathrm{p}_3') i\mathcal{A}^{J\lambda_{32}}_{\lambda_1;\lambda_2',\lambda_3';\lambda}(\Omega_1^{lab},\mathrm{p}_3',M_{23},\Omega_3^{cm})\\
&\equiv i\mathcal{M}^Z_{\lambda_2,\lambda_3;\lambda}(\Omega_1^{lab},M_{23},\Omega_3^{cm}),\\
i\mathcal{M}^Z_{\lambda_2;\lambda_1,\lambda_3;\lambda}(p_1,p_2,p_3) &= \sum_J N_J\sum_{\lambda_1'\lambda_3'}\int\frac{\tilde{\mathrm{p}}_3'^2 d\tilde{\mathrm{p}}_3'}{(2\pi)^3} i\mathcal{T}^J_{\lambda_1,\lambda_3;\lambda_1',\lambda_3'}(\tilde{\mathrm{p}}_3',M_{13})G_0(\tilde{\mathrm{p}}_3') i\mathcal{A}^{J\lambda_{31}}_{\lambda_2;\lambda_1',\lambda_3';\lambda}(\tilde{\Omega}_3^{lab},\tilde{\mathrm{p}}_3',M_{13},\tilde{\Omega}_3^{cm})\\
&\equiv i\mathcal{M}^Z_{\lambda_2;\lambda_1,\lambda_3;\lambda}(\tilde{\Omega}_3^{lab},M_{13},\tilde{\Omega}_3^{cm}).
\end{aligned}\tag{32}$$

Note: when consider the rescattering of different particles, the different cm frames should be adopted.

Decay width

The case with one rescattering

Phase space

The phase space is given by

$$d\Phi = (2\pi)^4\delta^4(P-p_1-p_2-p_3)\frac{d^3p_1}{2E_1(2\pi)^3}\frac{d^3p_2}{2E_2(2\pi)^3}\frac{d^3p_3}{2E_3(2\pi)^3}\tag{33}$$

To study the invariant mass spectrum of the particles 2 and 3, it is convenient to rewrite the Lorentz-invariant phase space $d\Phi$ by taking as integration variables the direction of the momentum of particle 2 $\boldsymbol{p}_2^{cm}$ in the center-of-mass (cm) frame of particles 2 and 3. Thus, we first rewrite the phase factor as

$$d\Phi = (2\pi)^4\delta(E_2^{cm}+E_3^{cm}-W_{23})\delta^3(\boldsymbol{p}_2^{cm}+\boldsymbol{p}_3^{cm})\frac{d^3p_1^{lab}}{2E_1^{lab}(2\pi)^3}\frac{d^3p_2^{cm}}{2E_2^{cm}(2\pi)^3}\frac{d^3p_3^{cm}}{2E_3^{cm}(2\pi)^3}\tag{34}$$

where $W_{23}^2=(M-E_1^{lab})^2-|\mathrm{p}_1^{lab}|^2$. Here the Lorentz invariance of the $\frac{d^3p}{2E(2\pi)^3}$ and $\delta^4(P-p_1-p_2-p_3)$ is used.

Here, we use the momentum of the particle 3 in the center of mass system of two rescattering particles.
The momentum of the particle 3 has a relation $\mathrm{p}_3^{cm}=\frac{1}{2M_{23}}\sqrt{\lambda(M_{23}^2,m_3^2,m_2^2)}$.

Owing to the three-momentum δ function, the integral over $\boldsymbol{p}_2^{cm}$ can be eliminated. Next, the quantity $d^3p_3^{cm}$ is converted to dM_{23} by the relation,

$$d^3p_3^{cm}=\frac{E_2^{cm}E_3^{cm}\mathrm{p}_3^{cm}}{M_{23}}dM_{23}d\Omega_3^{cm}, \tag{35}$$

where $M_{23}(=E_2^{cm}+E_3^{cm})$ is the invariant mass of the 23 system. We would like to integrate over the magnitude of the neutron momentum p_1, which is related to W_{23}. Hence, the energy-conserving δ function is substituted as,

$$\delta(M_{23}-W_{23})=\frac{W_{23}}{|M\mathrm{p}_1^{lab}/E_1^{lab}|}\delta(\check{\mathrm{p}}_1-\mathrm{p}_1^{lab}) \tag{36}$$

where the $\check{\mathrm{p}}_1$ satisfies $M_{23}^2=(M-\check{E}_1)^2-\check{\mathrm{p}}_1^2$.

Performing the integral over p_1^{lab}, we obtain the final expression of the decay width,

$$d\Phi=\frac{1}{(2\pi)^5}\frac{\check{\mathrm{p}}_1\mathrm{p}_3^{cm}}{8M}d\Omega_1^{lab}d\Omega_3^{cm}dM_{23}=\frac{1}{(2\pi)^3}\frac{\check{\mathrm{p}}_1\mathrm{p}_3^{cm}}{8M}d\cos\theta_1^{lab}d\cos\theta_3^{cm}dM_{23}, \tag{37}$$

Here, independence of the ϕ_1^{lab} and ϕ_2^{cm} on the integrand is applied.

Differential decay width

Here, we consider the rescattering of particles 2 and 3. The distribution can be obtained as

$$\frac{d\Gamma}{dM_{23}}=\int\frac{1}{2M}\sum_{\lambda_2,\lambda_3;\lambda}|i\mathcal{M}_{\lambda_2,\lambda_3;\lambda}|^2\frac{1}{(2\pi)^5}\frac{\check{\mathrm{p}}_1\mathrm{p}_3}{8M}d\Omega_1^{lab}d\Omega_3^{cm} \tag{38}$$

If the Y is not scalar, there shoold be an additional factor $1/\tilde{j}_Y$. Here we
use the distribution above in the frame with Lorentz boost but without rotation. However,
we calculate the $|i\mathcal{M}_{\lambda_2,\lambda_3;\lambda}|^2$ in the frame with both Lorentz boost and rotation due to it is independent on the frames.

If the reflection effect is absent, for example, the πDD^* final state. The results can be simplified further.
The amplitude can be written as

$$\begin{aligned}i\mathcal{M}_{\lambda_1\lambda_2,\lambda_3;\lambda}(p_1,p_2,p_3)&=\sum_{JM}N_J^3D^J_{\lambda_R\lambda_1}D^J_{\lambda_R\lambda_{32}}(\Omega_3^{cm})\Big[i\mathcal{A}^{d,J}_{\lambda_1,\lambda_2,\lambda_3;\lambda}(M_{23})\\&\quad+\sum_{\lambda_2'\lambda_3'}\int\frac{\mathrm{p}_3'^2}{(2\pi)^3}i\mathcal{T}^J_{\lambda_2,\lambda_3;\lambda_2',\lambda_3'}(\mathrm{p}_3',M_{23})G_0(\mathrm{p}_3')i\mathcal{A}^J_{\lambda_2',\lambda_3';\lambda}(\mathrm{p}_3',M_{23})\Big]\\&\equiv\sum_{JM}N_J^3D^J_{\lambda_R\lambda_1}(\Omega_1^{lab})D^J_{\lambda_R\lambda_{32}}(\Omega_3^{cm})i\mathcal{M}^J_{\lambda_2,\lambda_3;\lambda}(M_{23})\end{aligned} \tag{39}$$

Inserting the above amplitude to the definition of the invariant mass spectrum, we have

$$\begin{aligned}\frac{d\Gamma}{dM_{23}}&=\frac{1}{2M}\frac{1}{(2\pi)^5}\frac{\check{\mathrm{p}}_1\mathrm{p}_3}{8M}\sum_{\lambda_2,\lambda_3;\lambda}\int|\sum_{JM}N_J^3D^J_{\lambda_R\lambda_1}(\Omega_1^{lab})D^J_{\lambda_R\lambda_{32}}(\Omega_3^{cm})i\mathcal{M}^J_{\lambda_2,\lambda_3;\lambda}(M_{23})|^2d\Omega_1^{lab}d\Omega_3\\&=\frac{1}{2M}\frac{1}{(2\pi)^5}\frac{\check{\mathrm{p}}_1\mathrm{p}_3^{cm}}{8M}\sum_{\lambda_2,\lambda_3;\lambda;J}|N_Ji\mathcal{M}^J_{\lambda_2,\lambda_3;\lambda}(M_{23})|^2\end{aligned}$$

Now we consider the amplitude with fixed parity,

$$i\mathcal{M}^J_{\lambda_{23};\lambda} = i\mathcal{A}^{d,J}_{\lambda_{23};\lambda} + i\mathcal{T}^J_{\lambda_{23},\lambda'_{23}} G_0 i\mathcal{A}^J_{\lambda'_{23};\lambda}, \quad \eta i\mathcal{M}^J_{-\lambda_{23};\lambda} = \eta i\mathcal{A}^{d,J}_{-\lambda_{23};\lambda} + \eta i\mathcal{T}^J_{-\lambda_{23},\lambda'_{23}} G_0 i\mathcal{A}^J_{\lambda'_{23};\lambda}$$
$$\Rightarrow i\mathcal{M}^{J^P}_{\lambda_{23};\lambda} = i\mathcal{A}^{d,J^P}_{\lambda_{23};\lambda} + i\mathcal{T}^{J^P}_{\lambda_{23},\lambda'_{23}} G_0 i\mathcal{A}^J_{\lambda'_{23};\lambda} = i\mathcal{A}^{d,J^P}_{\lambda_{23};\lambda} + \eta' i\mathcal{T}^{J^P}_{\lambda_{23},\lambda'_{23}} G_0 i\mathcal{A}^J_{-\lambda'_{23};\lambda} = i\mathcal{A}^{d,J^P}_{\lambda_{23};\lambda} + \frac{1}{2} i\mathcal{T}^{J^P}_{\lambda_{23},\lambda'_{23}} G_0 i\mathcal{A}^{J^P}_{\lambda'_{23};\lambda} \tag{40}$$

Here $\mathcal{M}^{J^P}_{\lambda_{23};\lambda} = \mathcal{M}^J_{\lambda_{23};\lambda} + \eta\mathcal{M}^J_{\lambda_{23};\lambda}$, $\mathcal{A}^{d,J^P}_{\lambda_{23};\lambda} = \mathcal{A}^{d,J}_{\lambda_{23};\lambda} + \eta\mathcal{A}^{d,J}_{\lambda_{23};\lambda}$, $\mathcal{T}^{J^P}_{\lambda_{23},\lambda'_{23}} = \mathcal{T}^J_{\lambda_{23},\lambda'_{23}} + \eta\mathcal{T}^J_{-\lambda_{23},\lambda'_{23}} = \mathcal{T}^J_{\lambda_{23},\lambda'_{23}} + \eta'\mathcal{T}^J_{\lambda_{23},-\lambda'_{23}}$.
NOTE: For parity conserving interaction, we have $\mathcal{T}^J_{\lambda'\lambda} = \eta(\eta')^{-1}\mathcal{T}_{-\lambda'\lambda}$, which can be checked in the code by different definitions of $\mathcal{T}^{J^P}$.

We summarize the results as following,

$$\frac{d\Gamma}{dM_{23}} = \frac{1}{2M}\frac{1}{(2\pi)^5}\frac{\breve{\mathrm{p}}_1\mathrm{p}_3}{8M}\sum_{ij;J^P}|N_J\mathcal{M}^{J^P}_{ij}(M_{23})|^2,$$
$$\hat{\mathcal{M}}^{J^P}_{i;i}(M_{23}) = \mathcal{A}^{d,J^P}_{j;i}(M_{23}) + \sum_k\int\frac{d\mathrm{p}'_3\mathrm{p}'^2_3}{(2\pi)^3}\mathcal{T}^{J^P}_{j;k}(\mathrm{p}'_3, M_{23})G_0(\mathrm{p}'_3)\mathcal{A}^{J^P}_{k;i}(\mathrm{p}'_3, M_{23}), \tag{41}$$

where i and j denote the independent $\lambda_{2,3}$ and λ, and the factors $f_{i=0} = 1/\sqrt{2}$ and $f_{i\neq 0} = 1$ are inserted. The above equation can be abbreviated as $M = A^d + TGA$, where T is solved by the Bethe-Salpeter equation $T = V + VGT$. NOTE: The $\hat{T}$ should be multiplied by 4π to be used as $\mathcal{T}$, and $\hat{G}_0$ should be divided by 4π to be used as G_0, which is cancelled by each other.

The case with more than one rescattering

In the case with more than one resacttering, we should conisder Monte-Carlo method to generate the event.

$$d\Gamma = \frac{1}{2E}\sum|\mathcal{M}|^2 d\Phi = \frac{1}{2E}\sum|\mathcal{M}|^2(2\pi)^{4-3n}dR \tag{42}$$

Here we consider a process with direct, 23 rescattering, 13 rescattering.

$$i\mathcal{M}^d_{\lambda_1;\lambda_2,\lambda_3;\lambda}(p_1,p_2,p_3) = i\mathcal{A}_{\lambda_1;\lambda_2,\lambda_3;\lambda}(p_1,p_2,p_3),$$
$$i\mathcal{M}^Z_{\lambda_k;\lambda_i,\lambda_j;\lambda}(p_k,p_i,p_j) = \sum_J N_J\sum_{\lambda'_i\lambda'_j}\int\frac{\mathrm{p}'^2_j d\mathrm{p}'_j}{(2\pi)^3} i\mathcal{T}^J_{\lambda_i,\lambda_j;\lambda'_i,\lambda'_j}(\mathrm{p}'_j, M_{ij})\, G_0(\mathrm{p}'_j) i\mathcal{A}^J_{\lambda_k;\lambda'_i,\lambda'_j;\lambda}(\Omega^{lab}_k, \mathrm{p}'_j, M_{ij}, \Omega^{cm}_j) \tag{43}$$

$$\begin{aligned}\sum_{\lambda'}T^J_{\lambda\lambda'}A^J_{\lambda'} &= T^J_{\lambda 0}A^J_0 + \sum_{j>0}\left[T^J_{\lambda j}A^J_j + T^J_{\lambda -j}A^J_{-j}\right]\\ &= \frac{1}{4}T^{J^P}_{\lambda 0}A^{J^P}_0 + \sum_{j>0}\left[\frac{1}{4}(T^{J+}_{\lambda j} + T^{J-}_{\lambda j})(A^{J+}_j + A^{J-}_j) + \frac{1}{4}(T^{J+}_{\lambda j} - T^{J-}_{\lambda j})(A^{J+}_j - A^{J-}_j)\right]\\ &= \frac{1}{4}T^{J^P}_{\lambda 0}A^{J^P}_0 + \sum_{j>0}\left[\frac{1}{2}T^{J+}_{\lambda j}A^{J+}_j + \frac{1}{2}T^{J-}_{\lambda j}A^{J-}_j\right]\\ &= \frac{1}{2}\sum_P T^{J^P}_\lambda A^{J^P}\end{aligned}$$

$$\mathcal{A}^{J\lambda_{ji}}_{\lambda_k;\lambda'_i,\lambda'_j;\lambda}(\Omega^{lab}_k, \mathrm{p}'_j, M_{ij}, \Omega^{cm}_j) = N_J 2\pi\int d\cos\theta'_j d^J_{\lambda_{ji},\lambda'_{ji}}(\theta'_j)\mathcal{A}_{\lambda_k;\lambda'_i,\lambda'_j;\lambda}(\Omega^{lab}_k, \theta'_j, \mathrm{p}'_j, M_{ij}, \Omega^{cm}_j) \tag{44}$$

$$i\mathcal{M}^Z_{\lambda_k;\lambda_i,\lambda_j;\lambda}(p_k,p_i,p_j) = \frac{1}{2}\sum_{J^P}N_J\sum_{ij}\int\frac{\mathrm{p}'^2_j d\mathrm{p}'_j}{(2\pi)^3} i\mathcal{T}^{J^P}_{\lambda_i,\lambda_j;ij}(\mathrm{p}'_j, M_{ij})\, G_0(\mathrm{p}'_j) i\mathcal{A}^{J^P}_{\lambda_k;ij;\lambda}(\Omega^{lab}_k, \mathrm{p}'_j, M_{ij}, \Omega^{cm}_j) \tag{45}$$

$$\sum_{\lambda'}|\frac{1}{2}T^{J^P}_\lambda A^{J^P}|^2 = |\frac{1}{2}T^{J^P}_0 A^{J^P}|^2 + 2\sum_{j>0}|\frac{1}{2}T^{J^P}_j A^{J^P}|^2 = \frac{1}{2}\sum_j|T^{J^P}A^{J^P}|^2 \tag{46}$$

$$\sum_{\lambda_k;\lambda_i,\lambda_j;\lambda} |i\mathcal{M}^{Z}_{\lambda_k;\lambda_i,\lambda_j;\lambda}(p_k,p_i,p_j)|^2 = \frac{1}{2}\sum_{J^P,\lambda_k,\lambda} N_J^2 \sum_j |T^{J^P}A^{J^P}|^2 \tag{47}$$

- The GEN pacakge can generate the events with $(p_1^{lab}, p_2^{lab}, p_3^{lab})$ in the frame of Y.
- The M_{ij} can be obtained as $(p_i^{lab}+p_j^{lab})^2$, and used to solve the $\mathcal{T}^{J^P}_{\lambda_i,\lambda_j;\lambda'_i,\lambda'_j}(\mathrm{p}'_j, M_{ij})$ with qBSE. Here, the p'_j is generated by the Gauss discretization.
- With Lorentz boost and rotation, the momentum of p_i in the cernter of mass frame of partilces ij can be calculated. However, for the intermediate particles,the $D^J_{\lambda_{ji},\lambda'_{ji}}(\Omega'_j)$ is used to calculate $\mathcal{A}^{J\lambda_{ji}}_{\lambda_k;\lambda'_i,\lambda'_j;\lambda}(\Omega_k^{lab}, \mathrm{p}'_j, M_{ij})$.
- Because only $|\mathcal{M}|^2$ has the same value in different frames, the $\mathcal{M}$ for rescatterings of particles 23 and 13 can not be summed up directly.