

# Multiple scattering of waves

Artur L. Gower<sup>a</sup>,

<sup>a</sup> School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK

January 9, 2021

## Abstract

Here we show and deduce the T-matrix and a general multiple scattering formulation which can be adapted to acoustics, electromagnetism, and elasticity. For details on each specific physical medium see the other documents.

*Keywords:* Multiple scattering, T-matrix, Scattering matrix

## 1 Using a T-matrix

A T-matrix denotes how one single particle scatters waves [4, 3].

For convenience and generality we denote:

$$\begin{aligned} u_n(k\mathbf{r}) &= \text{outgoing spherical waves,} \\ v_n(k\mathbf{r}) &= \text{regular spherical waves,} \end{aligned} \tag{1}$$

where  $n$  denotes a multi index which depends on the dimension and if the waves are scalar or vector fields.

Any incident wave and scattered wave\*, centred at the same coordinate axis, can be written as

$$u_{\text{inc}} = \sum_n g_n v_n(k\mathbf{r}), \tag{2}$$

$$u_{\text{sc}} = \sum_n f_n u_n(k\mathbf{r}). \tag{3}$$

---

\*For the scattered wave we need only use outgoing spherical waves when measuring the field outside of a sphere which completely encompasses the particle.

The T-matrix is an infinite matrix such that

$$f_n = \sum_{n'} T_{nn'} g_{n'}. \quad (4)$$

Such a matrix  $T$  exists when scattering is a linear operation (elastic scattering).

We can also estimate the field inside the particle by assuming that the field is smooth and continuous. This approximation is exact for homogeneous spheres and cylinders, but not for a Circular cylindrical capsule.

Assume the field inside the particle can be described by a regular spherical series:

$$v_{\text{in}} = \sum_n b_n v_n(k_o \mathbf{r}), \quad (5)$$

where  $k_o$  is the particles wavenumber. Now if we assume that the total field is continuous everywhere so that  $u_{\text{inc}} + u_{\text{sc}} = v_{\text{in}}$  on the boundary of the particle. If the field was smooth enough, we could analytically extend the field  $v_{\text{in}}$  to a spherical boundary, with radius  $a$ , which contains the particle. Let's take this as an assumption and equate  $u_{\text{inc}} + u_{\text{sc}} = v_{\text{in}}$  for  $r = a$ . Due to orthogonality of the angular components of the basis functions this will result in

$$g_n v_n(k \mathbf{r}) + f_n u_n(k \mathbf{r}) = b_n v_n(k_o \mathbf{r}), \quad \text{for } |\mathbf{r}| = a \quad (6)$$

using the T-matrix we can then write  $g_n = T_{nm}^{-1} f_m$ , which substituted above leads to

$$b_n = \frac{1}{v_n(k_o \mathbf{r})} [v_n(k \mathbf{r}) T_{nm}^{-1} f_m + u_n(k \mathbf{r}) f_n], \quad \text{for } |\mathbf{r}| = a. \quad (7)$$

## 2 Multiple scattering in general

For multiple scattering in higher dimensions and for vector wave equations we use the notation given in [5].

For a point  $\mathbf{r}$ , outside of the circumscribed spheres of all particles, we can write the total field  $u(\mathbf{r})$  as a sum of the incident wave  $u_{\text{inc}}(\mathbf{r})$  and all scattered waves in the form [6, 7, 8]

$$u(\mathbf{r}) = u_{\text{inc}}(\mathbf{r}) + u_{\text{sc}}(\mathbf{r}), \quad u_{\text{sc}}(\mathbf{r}) = \sum_{i=1}^N \sum_n f_n^i u_n(k \mathbf{r} - k \mathbf{r}_i), \quad (8)$$

where we assumed  $|\mathbf{r} - \mathbf{r}_i| > a_i$  for  $i = 1, 2, \dots, N$ , the  $f_n^i$  are coefficients we need to determine, where again:

$$\begin{cases} u_n(k \mathbf{r}) = \text{outgoing spherical waves,} \\ v_n(k \mathbf{r}) = \text{regular spherical waves,} \end{cases} \quad (9)$$

where  $n$  denotes a multi index which depends on the dimension and if the waves are scalar or vector fields.

In general, we can write the multiple scattering system in the form:

$$\alpha_n^i = g_n^i + \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{n' n''} \mathcal{U}_{n'' n}(k\mathbf{r}_i - k\mathbf{r}_j) T_{n'' n'}^j \alpha_{n'}^j, \quad (10)$$

for  $i = 1, 2, \dots, N$ , where  $f_n^i = \sum_{n'} T_{nn'}^i \alpha_{n'}^i$  and  $\mathcal{U}_{nn'}$  is a translation matrix [1, 2]. Let  $\mathbf{r}' = \mathbf{r} + \mathbf{d}$ , then the translation matrices for a translation  $\mathbf{d}$  can be defined by the property [1]

$$\begin{cases} \mathbf{v}_n(k\mathbf{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\mathbf{d}) \mathbf{v}_{n'}(k\mathbf{r}), & \text{for all } \mathbf{d} \\ \mathbf{u}_n(k\mathbf{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\mathbf{d}) \mathbf{u}_{n'}(k\mathbf{r}), & |\mathbf{r}| > |\mathbf{d}| \\ \mathbf{u}_n(k\mathbf{r}') = \sum_{n'} \mathcal{U}_{nn'}(k\mathbf{d}) \mathbf{v}_{n'}(k\mathbf{r}), & |\mathbf{r}| < |\mathbf{d}| \end{cases} \quad (11)$$

## 2.1 Turing equations into code

For easy implementation we need the functions:

$$\psi_{\text{inc}} \mapsto g_j^m \quad \text{and} \quad \text{particle} \mapsto T_j^{nm}.$$

For efficient implementation we rewrite (10) as a matrix equation. Let

$$(\boldsymbol{\alpha}_j)_n = \alpha_n^j, \quad (\mathbf{g}_j)_n = g_n^j, \quad (12)$$

$$(\mathbf{T}_j)_{nn'} = T_{nn'}^j, \quad (\mathbf{U}_{j\ell})_{n'n} = \mathcal{U}_{n'n}(k\mathbf{r}_j - k\mathbf{r}_\ell), \quad (13)$$

Then

$$\sum_{\ell} (\delta_{j\ell} + (\delta_{j\ell} - 1) \mathbf{U}_{j\ell}^T \mathbf{T}_\ell) \boldsymbol{\alpha}_\ell = \mathbf{g}_j, \quad (14)$$

where  $\cdot^T$  is the transpose operation. The above then leads to a block matrix equation:

$$\begin{bmatrix} \mathbf{I} & -\mathbf{U}_{12}^T \mathbf{T}_2 & \cdots & -\mathbf{U}_{1(N-1)}^T \mathbf{T}_{N-1} & -\mathbf{U}_{1N}^T \mathbf{T}_N \\ -\mathbf{U}_{21}^T \mathbf{T}_1 & \mathbf{I} & -\mathbf{U}_{23}^T \mathbf{T}_3 & \cdots & -\mathbf{U}_{2N}^T \mathbf{T}_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{U}_{N1}^T \mathbf{T}_1 & \cdots & \cdots & -\mathbf{U}_{N(N-1)}^T \mathbf{T}_{N-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_N \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_N \end{bmatrix} \quad (15)$$

## References

- [1] A. Boström, G. Kristensson, and S. Ström. “Transformation Properties of Plane, Spherical and Cylindrical Scalar and Vector Wave Functions”. In: *Field Representations and Introduction to Scattering*. Ed. by V. V. Varadan, A. Lakhtakia, and V. K. Varadan. Acoustic, Electromagnetic and Elastic Wave Scattering. 1991. Chap. 4, pp. 165–210.
- [2] B. Friedman and J. Russek. “Addition theorems for spherical waves”. In: 12 (1954), pp. 13–23.
- [3] M. Ganesh and S. C. Hawkins. “Algorithm 975: TMatROM—A T-Matrix Reduced Order Model Software”. In: *ACM Trans. Math. Softw.* 44 (July 2017), 9:1–9:18. (Visited on 03/23/2018).
- [4] Mahadevan Ganesh and Stuart Collin Hawkins. “A far-field based T-matrix method for two dimensional obstacle scattering”. In: *ANZIAM Journal* 51 (May 12, 2010), pp. 215–230. (Visited on 03/23/2018).
- [5] Artur Lewis Gower and Gerhard Kristensson. “Effective Waves for Random Three-dimensional Particulate Materials”. In: *arXiv preprint arXiv:2010.00934* (2020).
- [6] G. Kristensson. “Coherent scattering by a collection of randomly located obstacles — an alternative integral equation formulation”. In: 164 (2015), pp. 97–108.
- [7] G. Kristensson. *Scattering of Electromagnetic Waves by Obstacles*. Mario Boella Series on Electromagnetism in Information and Communication. Edison, NJ, USA: SciTech Publishing, 2016.
- [8] C. M. Linton and P. A. Martin. “Multiple Scattering by Multiple Spheres: A New Proof of the Lloyd–Berry Formula for the Effective Wavenumber”. In: 66 (2006), pp. 1649–1668.