

These notes for the New Keynesian model follow Eric Sims's notes. Section 1 solves the equilibrium conditions of the New Keynesian model, and Section 2 transforms the equilibrium conditions into the desired form for a risk-adjusted linearization.

# 1 Model

## 1.1 Household

Households solve the problem

$$\max_{C_t, N_t, B_{t+1}, M_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log \left( \frac{M_t}{P_t} \right) \right)$$

subject to the budget constraint

$$P_t C_t + B_{t+1} + M_t - M_{t-1} \leq W_t N_t + \Pi_t + (1 + i_{t-1}) B_t.$$

In this model, households have demand for money  $M_t$ , which is also the numeraire. The price of goods in terms of money is  $P_t$ . The stock of nominal bonds a households has is  $B_t$ . Note that  $B_t$  will be pre-determined at period  $t$  while  $M_t$  will not be ( $M_{t-1}$  is pre-determined).

The Lagrangian for the household is

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log \left( \frac{M_t}{P_t} \right) \right] \\ & + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [\lambda_t (P_t C_t + B_{t+1} + M_t - M_{t-1} - W_t N_t + \Pi_t + (1 + i_{t-1}) B_t)], \end{aligned}$$

which implies first-order conditions

$$\begin{aligned} 0 &= C_t^{-\sigma} - \lambda_t P_t \\ 0 &= -\psi N_t^{\eta} + \lambda_t W_t \\ 0 &= -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} (1 + i_t) \\ 0 &= \theta \frac{1}{M_t} - \lambda_t + \beta \mathbb{E}_t \lambda_{t+1}. \end{aligned}$$

The first two equations can be combined by isolating  $\lambda_t$ . Using  $\lambda_t = C_t^{-\sigma} / P_t$ , we can obtain the Euler equation for households and an equation relating money balances to consumption.

$$\begin{aligned} C_t^{-\sigma} \frac{W_t}{P_t} &= \psi N_t^{\eta}, \\ C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma} (1 + i_t), \\ \theta \left( \frac{M_t}{P_t} \right)^{-1} &= \frac{i_t}{1 + i_t} C_t^{-\sigma}. \end{aligned}$$

## 1.2 Production

**Final Producers** There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

where  $\epsilon > 1$  so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj.$$

The FOC for  $Y_t(j)$  is

$$\begin{aligned} 0 &= P_t \frac{\epsilon}{\epsilon-1} \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j) \\ 0 &= \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t} \\ 0 &= \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \\ Y_t(j) &= \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t. \end{aligned}$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) dj$$

and simplifying yields the price index

$$P_t = \left( \int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}.$$

**Intermediate Producers** Intermediate goods are producing according to the linear technology

$$Y_t(j) = A_t N_t(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{N_t(j)} W_t N_t(j) \quad \text{s.t.} \quad A_t N_t(j) \geq \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The Lagrangian is

$$\mathcal{L} = W_t N_t(j) + \varphi_t(j) \left( \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - A_t N_t(j) \right),$$

so the first-order condition is

$$0 = W_t - \varphi_t(j) A_t \Rightarrow \varphi_t(j) = \frac{W_t}{A_t}.$$

The multiplier  $\varphi_t$  can be interpreted as the nominal marginal cost. Let  $mc_t$  be the real marginal cost. Then profits for an intermediate producer is

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j).$$

In addition to the labor choice, firms also have the chance to reset prices in every period with probability  $1 - \phi$ . This problem can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} \left( \frac{P_t(j)}{P_{t+s}} \left( \frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left( \frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right),$$

where I have imposed that output equals demand. The first-order condition is

$$\begin{aligned} 0 &= (1 - \epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s} \\ &\quad + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s} \end{aligned}$$

Divide by  $P_t(j)^{-\epsilon}/u'(C_t)$  and re-arrange to obtain

$$P_t(j) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) P_{t+s}^{\epsilon-1} Y_{t+s}}.$$

This expression gives the optimal reset price  $P_t^*$ , which we can write more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{X_{1,t}}{X_{2,t}}$$

where

$$\begin{aligned} X_{1,t} &= u'(C_t) mc_t P_t^{\epsilon} Y_t + \phi \beta \mathbb{E}_t X_{1,t+1} \\ X_{2,t} &= u'(C_t) P_t^{\epsilon-1} Y_t + \phi \beta \mathbb{E}_t X_{2,t+1}. \end{aligned}$$

### 1.3 Equilibrium and Aggregation

To close the model, I assume that the log of technology  $A_t$  follows the AR(1)

$$\log A_t = \rho_a \log A_{t-1} + \varepsilon_{a,t},$$

and the growth rate in the log money supply follows the AR(1)

$$\Delta \log M_t = (1 - \rho_m)\pi + \rho_m \Delta \log M_{t-1} + \varepsilon_{m,t},$$

where  $\pi$  is the steady-state rate of inflation. Note that this specification ensures that money balances grow at the same rate as the price level, which ensures real balances are stationary. To re-write the money growth equation in real terms, note that

$$\log(m_t) = \log(M_t) - \log(P_t) \Rightarrow \Delta \log(m_t) = \log(m_t) - \log(m_{t-1}) = \Delta \log(M_t) - \log(1 + \pi_t),$$

hence

$$\Delta \log(m_t) = (1 - \rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1 + \pi_{t-1}) - \log(1 + \pi_t) + \varepsilon_{m,t}.$$

In equilibrium, bond-holding must be zero, hence

$$C_t = w_t N_t + \frac{\Pi_t}{P_t}.$$

Real dividends  $\Pi_t$  satisfy the accounting identity

$$\begin{aligned} \frac{\Pi_t}{P_t} &= \int_0^1 \left( \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) \right) dj \\ &= \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t \int_0^1 N_t(j) dj \end{aligned}$$

where  $w_t = W_t/P_t$ . Aggregate labor supply  $N_t$  equals aggregate labor demand in equilibrium, and market-clearing for consumption requires

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj = \int_0^1 \frac{P_t(j)}{P_t} \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj = P_t^{\epsilon-1} Y_t \int_0^1 P_t(j)^{1-\epsilon} dj = Y_t$$

since  $\int_0^1 P_t(j)^{1-\epsilon} dj = P_t^{1-\epsilon}$ .

The quantity  $Y_t$  is aggregate output, so we must have

$$\begin{aligned} \int_0^1 A_t N_t(j) dj &= \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj \\ A_t N_t &= Y_t \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} dj = v_t Y_t. \end{aligned}$$

Thus, aggregate output is

$$Y_t = \frac{A_t N_t}{v_t}.$$

It can be shown that  $v_t \geq 1$  by applying Jensen's inequality. Finally, recall that

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj.$$

In each period, a fraction  $\phi$  cannot change their price. Without loss of generality, we may re-order these firms to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon} dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \phi \int_0^1 P_{t-1}(j)^{1-\epsilon} dj = (1-\phi)(P_t^*)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Dividing by  $P_{t-1}^{1-\epsilon}$  implies

$$(1 + \pi_t)^{1-\epsilon} = (1-\phi)(1 + \pi_t^*)^{1-\epsilon} + \phi$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$\begin{aligned} v_t &= \int_0^{1-\phi} \left( \frac{P_t^*}{P_t} \right)^{-\epsilon} dj + \int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_t} \right)^{-\epsilon} dj \\ &= \int_0^{1-\phi} \left( \frac{P_t^*}{P_{t-1}} \right)^{-\epsilon} \left( \frac{P_{t-1}}{P_t} \right)^{-\epsilon} dj + \int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} \left( \frac{P_{t-1}}{P_t} \right)^{-\epsilon} dj \\ &= (1-\phi)(1 + \pi_t^*)^{-\epsilon}(1 + \pi_t)^{\epsilon} + (1 + \pi_t)^{\epsilon} \int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj. \end{aligned}$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj = \phi \int_0^1 \left( \frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj = \phi v_{t-1}.$$

Thus, we acquire

$$\begin{aligned} v_t &= (1-\phi)(1 + \pi_t^*)^{-\epsilon}(1 + \pi_t)^{\epsilon} + \phi(1 + \pi_t)^{\epsilon} v_{t-1} \\ &= (1 + \pi_t)^{\epsilon} ((1-\phi)(1 + \pi_t^*)^{-\epsilon} + \phi v_{t-1}). \end{aligned}$$

To finish, we need to derive an expression characterizing  $\pi_t^*$ . Define

$$x_{1,t} \equiv \frac{X_{1,t}}{P_t^\epsilon}, \quad x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon-1}}.$$

It follows that

$$\begin{aligned} x_{1,t} &= u'(C_t)mc_t Y_t + \phi\beta\mathbb{E}_t \frac{X_{1,t+1}}{P_t^\epsilon} \\ &= u'(C_t)mc_t Y_t + \phi\beta\mathbb{E}_t \left[ \frac{X_{1,t+1}}{P_{t+1}^\epsilon} \frac{P_{t+1}^\epsilon}{P_t^\epsilon} \right] \\ &= u'(C_t)mc_t Y_t + \phi\beta\mathbb{E}_t [x_{1,t+1}(1 + \pi_{t+1})] \\ x_{2,t} &= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t \frac{X_{2,t+1}}{P_t^{\epsilon-1}} \\ &= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t \left[ \frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}} \frac{P_{t+1}^{\epsilon-1}}{P_t^{\epsilon-1}} \right] \\ &= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t [x_{2,t+1}(1 + \pi_{t+1})]. \end{aligned}$$

Further,

$$\frac{X_{1,t}}{X_{2,t}} = \frac{x_{1,t}}{x_{2,t}} P_t,$$

hence

$$\begin{aligned} P_t^* &= \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} P_t \\ (1 + \pi_t^*) &= \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} (1 + \pi_t) \end{aligned}$$

All together, the full set of equilibrium conditions are

$$\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t \left[ C_{t+1}^{-\sigma} \frac{(1+i_t)}{1+\pi_{t+1}} \right] \\
C_t^{-\sigma} &= \psi \frac{N_t^\eta}{w_t} \\
m_t &= \theta \frac{1+i_t}{i_t} C_t^\sigma \\
mc_t &= \frac{w_t}{A_t} \\
C_t &= Y_t \\
Y_t &= \frac{A_t N_t}{v_t} \\
v_t &= (1+\pi_t)^\epsilon ((1-\phi)(1+\pi_t^*)^{-\epsilon} + \phi v_{t-1}) \\
(1+\pi_t)^{1-\epsilon} &= (1-\phi)(1+\pi_t^*)^{1-\epsilon} + \phi \\
(1+\pi_t^*) &= \frac{\epsilon}{\epsilon-1} \frac{x_{1,t}}{x_{2,t}} (1+\pi_t) \\
x_{1,t} &= C_t^{-\sigma} mc_t Y_t + \phi \beta \mathbb{E}_t [x_{1,t+1} (1+\pi_{t+1})] \\
x_{2,t} &= C_t^{-\sigma} Y_t + \phi \beta \mathbb{E}_t [x_{2,t+1} (1+\pi_{t+1})] \\
\log A_t &= \rho_a \log(A_{t-1}) + \varepsilon_{a,t} \\
\Delta \log(m_t) &= (1-\rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1+\pi_{t-1}) - \log(1+\pi_t) + \varepsilon_{m,t} \\
\Delta \log m_t &= \log m_t - \log m_{t-1},
\end{aligned}$$

which comprise 14 equations in 14 aggregate variables

$$(C_t, i_t, \pi_t, N_t, w_t, m_t, mc_t, A_t, Y_t, v_t, \pi_t^*, x_{1,t}, x_{2,t}, \Delta \log m_t).$$

Alternatively, the money growth equation can be replaced by the Taylor rule

$$\log(1+i_t) = (1-\rho_i) \log(1+i) + \rho_i \log(1+i_{t-1}) + (1-\rho_i) \phi_\pi (\log(1+\pi_t) - \log(1+\pi)) + \varepsilon_{i,t},$$

and the third equation relating money demand to consumption could also be ignored. To reduce the number of equations, we utilize this specification. Furthermore, we can also substitute  $1+\pi_t^*$  to remove  $\pi_t^*$  from the aggregate variables.

## 2 Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$\begin{aligned}
0 &= \log \mathbb{E}_t [\exp (\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1})] \\
z_{t+1} &= \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t) \varepsilon_{t+1},
\end{aligned}$$

where  $z_t$  are (predetermined) state variables and  $y_t$  are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and variables with a tilde are the logs of previously lower case variables (e.g. the real wage  $w_t$ )

The first equation becomes

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \left[ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1 + i_t}{1 + \pi_{t+1}} \right] \\ 0 &= \log \mathbb{E}_t \left[ \exp \left( \log(\beta) - \sigma(c_{t+1} - c_t) + \tilde{i}_t - \tilde{\pi}_{t+1} \right) \right] \\ &= \log \mathbb{E}_t \left[ \exp \left( \underbrace{\log(\beta) + \sigma c_t + \tilde{i}_t}_{\xi} - \underbrace{\sigma c_{t+1} - \tilde{\pi}_{t+1}}_{\text{Forward-Looking}} \right) \right], \end{aligned}$$

where  $c_t = \log(C_t)$ ,  $\tilde{i}_t = \log(1 + i_t)$ , and  $\tilde{\pi}_t = \log(1 + \pi_{t+1})$ .

The second equation becomes

$$\begin{aligned} 1 &= \psi \frac{N_t^\eta}{C_t^{-\sigma} w_t} \\ 0 &= \log \mathbb{E}_t \left[ \exp \left( \underbrace{\log(\psi) + \eta n_t - (-\sigma c_t + \hat{w}_t)}_{\xi} \right) \right], \end{aligned}$$

where  $n_t = \log(N_t)$  and  $\hat{w}_t = \log(w_t)$ .

The third equation becomes

$$\begin{aligned} 1 &= \frac{w_t}{A_t m c_t} \\ 0 &= \log \mathbb{E}_t \left[ \exp(\hat{w}_t - a_t - \tilde{m} c_t) \right]. \end{aligned}$$

The fourth and fifth equation become

$$0 = \log \mathbb{E}_t \left[ \exp(c_t - a_t - n_t + \hat{v}_t) \right].$$

The sixth equation becomes

$$0 = \hat{v}_t - \epsilon \tilde{\pi}_t - \log((1 - \phi) \exp(\tilde{\pi}_t^*)^{-\epsilon} + \phi \exp(\hat{v}_{t-1})),$$

where  $\hat{v}_{t-1}$  will be treated as an additional state variable, i.e. if  $a_t = \hat{v}_t$  is a jump variable and  $b_t = \hat{v}_{t-1}$  is a state variable, then

$$b_{t+1} = a_t.$$

The seventh equation becomes

$$0 = (1 - \epsilon) \tilde{\pi}_t - \log((1 - \phi) \exp(\tilde{\pi}_t^*)^{1-\epsilon} + \phi).$$



The eighth equation becomes

$$0 = \tilde{\pi}_t^* - \log\left(\frac{\epsilon}{\epsilon - 1}\right) - \tilde{\pi}_t - (\hat{x}_{1,t} - \hat{x}_{2,t}).$$

By plugging this expression for  $\tilde{\pi}_t^*$  into the previous two equations, we can also remove one more variable from the system.

The ninth and tenth equation become

$$\begin{aligned} 1 &= \mathbb{E}_t \left[ \phi \beta \frac{x_{1,t+1}(1 + \pi_{t+1})}{x_{1,t} - C_t^{-\sigma} m c_t A_t N_t / v_t} \right] \\ 0 &= \log \mathbb{E}_t \left[ \exp \left( \underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{1,t}) - \exp((1 - \sigma)c_t + \tilde{m}c_t))}_{\xi} + \hat{x}_{1,t+1} + \tilde{\pi}_{t+1} \right) \right] \\ 0 &= \log \mathbb{E}_t \left[ \exp \left( \underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{2,t}) - \exp((1 - \sigma)c_t))}_{\xi} + \hat{x}_{2,t+1} + \tilde{\pi}_{t+1} \right) \right], \end{aligned}$$

where the fact that  $x_{1,t}$  and  $x_{2,t}$  must both be positive implies  $x_{1,t} - C_t^{-\sigma} m c_t Y_t$  and  $x_{2,t} - C_t^{-\sigma} Y_t$  are both positive, as the expectations on the RHS are also both positive.

For the monetary policy rule, we use  $\tilde{i}_{t-1} \equiv \log(1 + i_{t-1})$  and  $\varepsilon_{i,t}$  as states and treat  $i_t$  as a jump variable, hence

$$\tilde{i}_t = (1 - \rho_i)\tilde{i} + \rho_i\tilde{i}_{t-1} + (1 - \rho_i)\phi_\pi(\tilde{\pi}_t - \tilde{\pi}) + \varepsilon_{i,t}.$$

This formulation allows us to treat the policy rule as an expectational equation.

The above nine equations comprise the expectational equations. The following four equations comprise the states:

$$\begin{aligned} a_{t+1} &= \rho_a a_t + \varepsilon_{a,t+1} \\ \hat{v}_{(t-1)+1} &= \hat{v}_t \\ \tilde{i}_{(t-1)+1} &= \tilde{i}_t \\ \varepsilon_{i,t+1} &= \varepsilon_{i,t+1} \end{aligned}$$