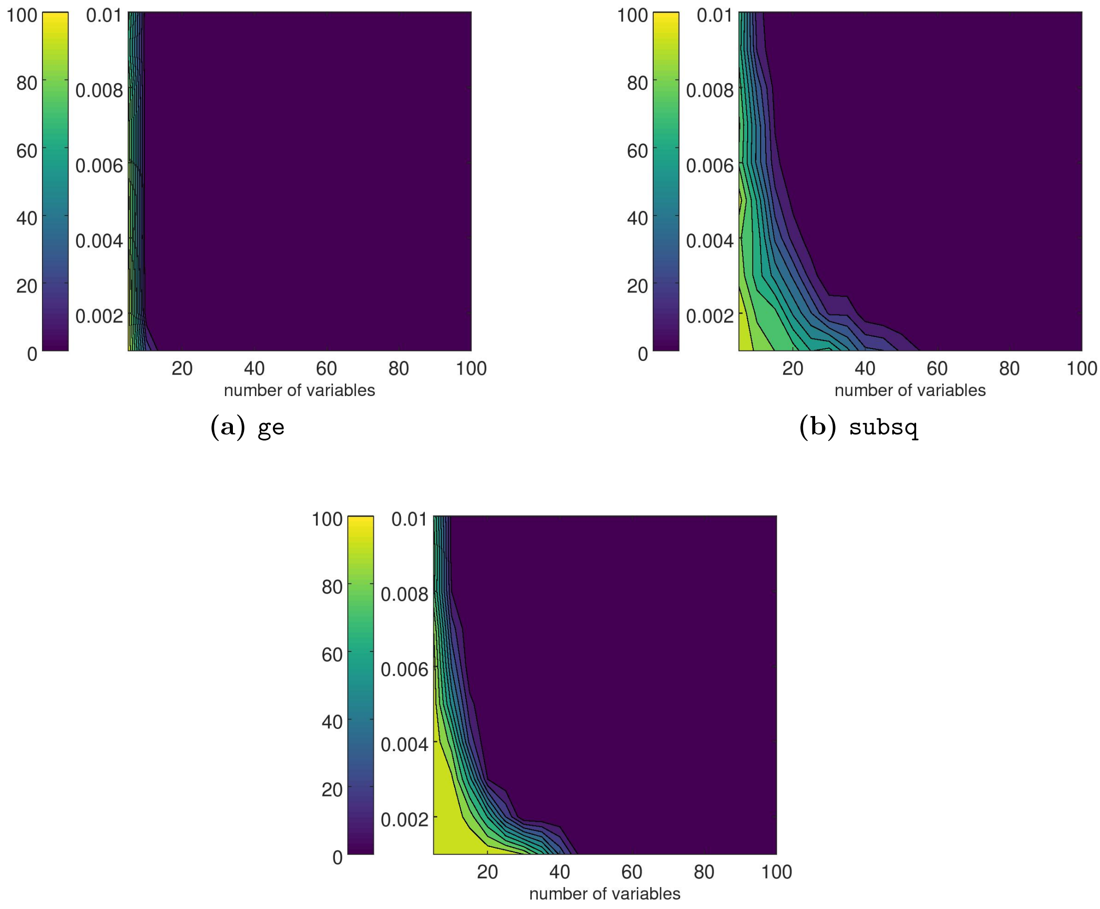
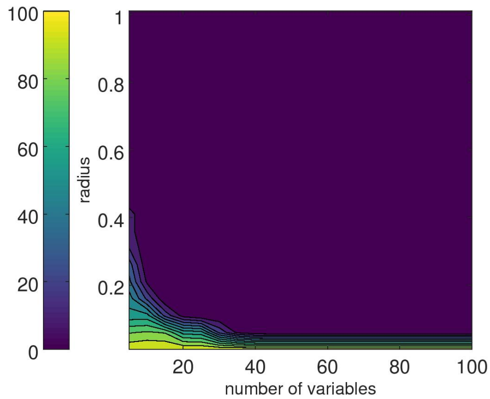
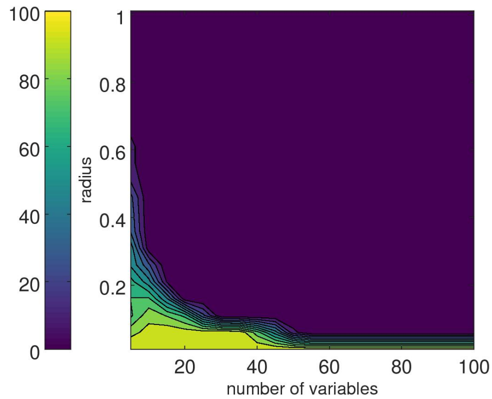


(a) ge (b) subsq (c) beeck

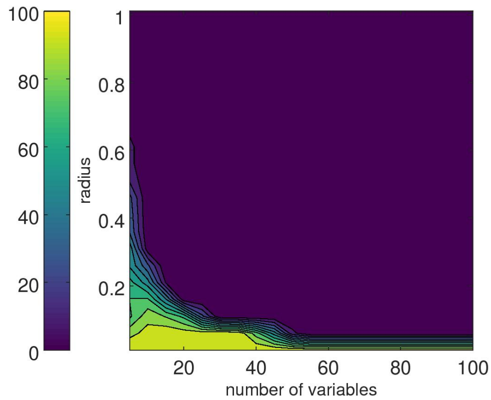
Figure 7.1: Strength of unsolvability tests `ge`, `subsq` and `beeck`. Color corresponds to percentage of unsolvable systems discovered (the lighter the area the higher the percentage).



(a) `fcr`

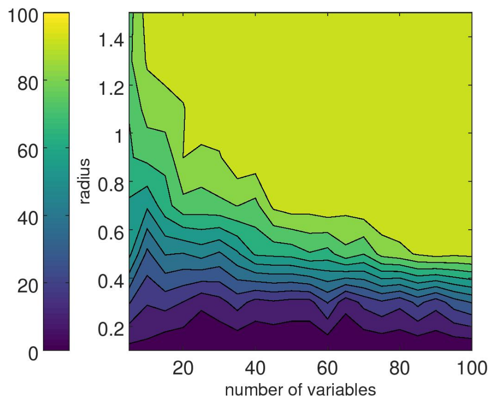


(b) `fcrit`

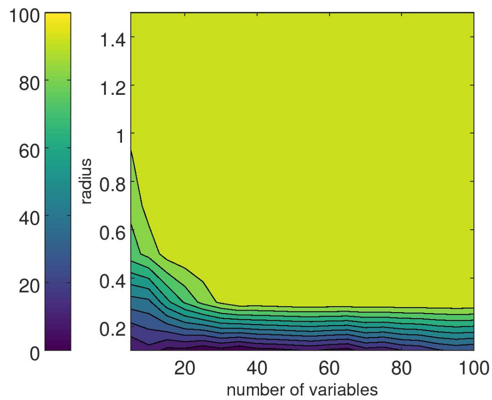


(c) `eig`

Figure 7.2: Strength of unsolvability tests `fcr`, `fcrit` and `eig`. Color corresponds to percentage of unsolvable systems discovered (the lighter the area the higher the percentage). Notice the different scale on y-axis in contrast to Figure 7.1.



(a) midpoint check



(b) orthant check

Figure 7.3: Strength of two solvability tests from Section 7.4. Color corresponds to percentage of solvable systems discovered (the lighter the area the higher percentage).

With growth of interval widths, generated systems become solvable. To check this we applied a similar test for the two solvability conditions. The results are depicted in 7.3. The orthant check is clearly better. The heat map (b) in Figure 7.2 and the heat map (b) in Figure 7.3 form a gap between the NP-complete and coNP-complete problem (unsolvability and solvability). For the tested systems the gap seems to be narrow.

8 Determinant of an interval matrix

- Known results
- Complexity of approximations
- Methods for computing determinant enclosures
- Determinant of symmetric matrices
- Classes of matrices with polynomially computable determinant bounds
- Comparison of methods

Applications of interval determinants were discussed in [208] for testing for Chebyshev systems or in [158] for computer graphics applications. Nevertheless, the area of interval determinants has not been much explored yet. In this chapter we address computational properties of determinants of general interval matrices. Next, we mention a known tool for computing interval determinants – interval Gaussian elimination. We then show how to modify existing tools from the classical linear algebra – Hadamard's inequality and the Gerschgorin circle theorem. After that, we design our new method based on Cramer's rule and solving interval linear systems. Regarding symmetric matrices, there are results about enclosing their eigenvalues that can also be used for computing interval determinants. All the methods work much better when combined with some kind of preconditioning. We briefly address this topic. Since computing a general interval determinant is intractable we point out classes of matrices with polynomially computable tasks connected to determinants. At the end we provide thorough testing of the mentioned methods on random general and symmetric interval matrices and discuss the use of these methods. The chapter is based on our work [86].

8.1 Definition

Definition 8.1 (Interval determinant 1)**.** Let $\boldsymbol{A}$ be a square interval matrix, then its determinant is defined as

$$\det(\boldsymbol{A}) = \{\det(A) \mid A \in \boldsymbol{A}\}.$$

Since the determinant of a real matrix is actually a polynomial, it is continuous. A closed interval is a compact set, so is the Cartesian product of them. Hence an interval matrix is a compact set. The image of the compact set under continuous mapping is again a compact set. That is why we can define the interval determinant in a more pleasant but equivalent way.

Definition 8.2 (Interval determinant 2)**.** Let $\boldsymbol{A}$ be a square interval matrix, then its determinant can be defined as the interval

$$\det(\boldsymbol{A}) = \Big[\min\{\det(A) \mid A \in \boldsymbol{A}\},\ \max\{\det(A) \mid A \in \boldsymbol{A}\}\Big].$$

Sometimes we refer to the exact determinant as the *hull*. In the following section we will state that computing the exact bounds on an interval determinant is an intractable problem. That is why, we are usually satisfied with an enclosure of the interval determinant. Of course, the tighter is the enclosure the better.

Definition 8.3 (Enclosure of interval determinant)**.** Let $\boldsymbol{A}$ be a square interval matrix, then an interval enclosure of its determinant is defined to be any $\boldsymbol{d} \in \mathbb{IR}$ such that

$$\det(\boldsymbol{A}) \subseteq \boldsymbol{d}.$$

Therefore, through this chapter we deal with the following problem:

Problem: Compute a tight enclosure of the determinant of $\boldsymbol{A}$.

8.2 Known results

To the best knowledge of ours, there are only a few theoretical results regarding interval determinants. Some of the results can be found in [112, 173]. In [173] we find a theorem stating that for an arbitrary matrix $A \in \boldsymbol{A}$ a matrix $A' \in \boldsymbol{A}$ can be found such that both A and A' have equal determinants and all coefficients of A', except one, come from some *edge matrix* of $\boldsymbol{A}$. (i.e., a real matrix where each coefficient A_{ij} is equal to the lower or upper bound of $\boldsymbol{A}_{ij}$).

Theorem 8.4 (Edge theorem)**.** *Let $\boldsymbol{A}$ be an interval matrix, then for each $A \in \boldsymbol{A}$, there exists a pair of indices (k,l) and $A' \in \boldsymbol{A}$ in the following form*

$$A'_{ij} \in \begin{cases} \{\underline{A}_{ij}, \overline{A}_{ij}\}, & (i,j) \neq (k,l), \\ \left[\underline{A}_{ij}, \overline{A}_{ij}\right], & (i,j) = (k,l), \end{cases}$$

such that $\det(A) = \det(A')$.

We prove the theorem with a more detailed demonstration than the one showed in [173].

Proof. Let $A \in \boldsymbol{A}$ be given. For a matrix $A' \in \boldsymbol{A}$ such that $\det(A) = \det(A')$, we remember the number of coefficients of A' such that $A'_{ij} \notin \{\underline{A}_{ij}, \overline{A}_{ij}\}$ (i.e., they do not lie on the edge of interval matrix). We wish to find A' that minimizes this number. We show that there exists $A' \in \boldsymbol{A}$ such that $\det(A') = \det(A)$ and this number is at most 1.

For the sake of contradiction let us assume that this number is 2 or greater. Thus there exist two pairs of indices $(p, q), (r, s)$ such that $A'_{pq} \in (\underline{A}_{pq}, \overline{A}_{pq})$ and $A'_{rs} \in (\underline{A}_{rs}, \overline{A}_{rs})$. Notice that here open intervals are used. The determinant of A' can be expressed as a function of these coefficients.

$$\det(A) = \det(A') = a \cdot A'_{pq} + b \cdot A'_{rs} + c \cdot A'_{pq}A'_{rs} + d, \tag{8.1}$$

for some $a, b, c, d \in \mathbb{R}$. When we fix the value of the determinant, we can express a variable (without loss of generality A'_{pq}) as

$$A'_{pq} = -\frac{b \cdot A'_{rs} + (d - \det(A))}{c \cdot A'_{rs} + a}, \tag{8.2}$$

which is a linear fractional function. Note that the denominator cannot be zero, otherwise it forces the function (8.1) to have only one variable which is a contradiction to our assumption that the number of variables is greater than or equal to 2.

The two cases, which are depicted in Figure 8.1, can occur. The dark box represents the Cartesian product of intervals $\boldsymbol{A}_{rs} \times \boldsymbol{A}_{pq}$. The first case represents a linear fractional function. In the second case the function degenerates to just a line. According to the definition of A' and (8.1) the point (A'_{pq}, A'_{rs}) lies in the interior of the box. Hence the function (8.2) intersects the box. We then move the point (A'_{pq}, A'_{rs}) along the graph of the function (8.2) to reach a new point (A''_{rs}, A''_{pq}) that lies on the border of the box. This way we actually obtained a new matrix A'' from A' that decreases the number of coefficients that do not belong to $\{\underline{A}_{ij}, \overline{A}_{ij}\}$ by one. If necessary, we can repeat the process and reduce the number of such coefficients to one.

□

The following claim is an immediate consequence and is also mentioned without an explicit proof in [173]. It claims that the exact bounds of the interval determinant can be computed as minimum and maximum determinant of all 2^{n^2} possible edge matrices of $\boldsymbol{A}$. Another reasoning for the corollary, not using the Edge theorem, is simply based just on linearity of determinant of a real matrix with respect to each coefficient.

Corollary 8.5. *For a given square interval matrix* $\boldsymbol{A}$ *the interval determinant can be obtained as*

$$\det(\boldsymbol{A}) = [\min(S), \max(S)], \ \textit{where } S = \{\det(A) \mid \forall i, j \ A_{ij} = \underline{A}_{ij} \textit{ or } A_{ij} = \overline{A}_{ij}\}.$$

Proof. For each $A \in \boldsymbol{A}$ a matrix A' can be constructed. This matrix has at most one coefficient $A'_{ij} \in (\underline{a}_{ij}, \overline{a}_{ij})$. A determinant of A' expressed in this coefficient is a linear

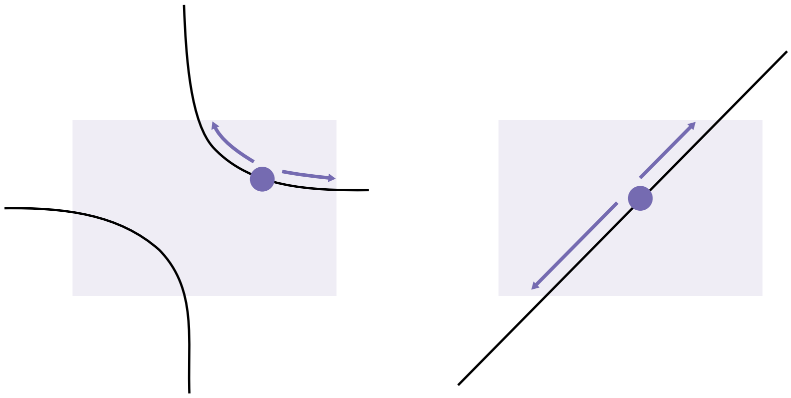


Figure 8.1: The two possible cases from the proof of Theorem 8.4. The dark box represents the Cartesian product of intervals $\boldsymbol{A}_{rs} \times \boldsymbol{A}_{pq}$. The curve represents the function (8.2).

function. Clearly the function value can be increased or decreased by setting

$$A'_{ij} = \underline{A}_{ij} \quad \text{or} \quad A_{ij} = \overline{A}'_{ij}.$$

That is why the matrix having minimum (or maximum) determinant must be some edge matrix of $\boldsymbol{A}$. □

A known result coming also from [173] is the following.

Theorem 8.6. *Let A_c be a rational nonnegative symmetric positive definite matrix. Then checking whether the interval matrix*

$$\boldsymbol{A} = [A_c - E, A_c + E]$$

is regular is a coNP*-complete problem.*

Proof. For a proof see, e.g., [173]. □

As a consequence of this theorem we can obtain the following important theorem [112, 173].

Theorem 8.7. *Let A_c be a rational nonnegative matrix. Computing the either of the exact bounds $\underline{\det(\boldsymbol{A})}$ or $\overline{\det(\boldsymbol{A})}$ of the matrix*

$$\boldsymbol{A} = [A_c - E, A_c + E]\,,$$

is NP*-hard.*

Proof. The proof of this theorem is also described in [112, 173]. □

8.3 Complexity of approximations

At the end of the previous section we stated that the problem of computing the exact bounds of the determinant of an interval matrix is generally an NP-hard problem. We could hope for having at least some approximation algorithms. Unfortunately, in this section we prove that this is not the case, neither for relative nor for absolute approximation.

Theorem 8.8 (Relative approximation, Horáček et al. [86])**.** *Let A_c be a rational nonnegative symmetric positive definite matrix. Let $\boldsymbol{A} = [A_c - E, A_c + E]$ and ε be arbitrary such that $0 < \varepsilon < 1$. If there exists a polynomial time algorithm returning $[\underline{a}, \overline{a}]$ such that*

$$\det(\boldsymbol{A}) \subseteq [\underline{a}, \overline{a}] \subseteq [1-\varepsilon, 1+\varepsilon] \cdot \det(\boldsymbol{A}),$$

then $\mathsf{P} = \mathsf{NP}$.

Proof. We use the fact from Theorem 8.6 that for a rational nonnegative symmetric positive definite matrix A_c, checking whether the interval matrix $\boldsymbol{A} = [A_c - E, A_c + E]$ is regular is a coNP-complete problem. We show that if such an ε-approximation algorithm existed, it would decide regularity from the above mentioned problem; which implies $\mathsf{P} = \mathsf{NP}$.

For a regular interval matrix we must have $\underline{\det(\boldsymbol{A})} > 0$ or $\overline{\det(\boldsymbol{A})} < 0$. If $\underline{\det(\boldsymbol{A})} > 0$ then, from the second inclusion $\underline{a} \geq (1-\varepsilon) \cdot \underline{\det(\boldsymbol{A})} > 0$. On the other hand, if $\underline{a} > 0$ then from the first inclusion $\underline{\det(\boldsymbol{A})} \geq \underline{a} > 0$. Therefore, we have $\underline{\det(\boldsymbol{A})} > 0$ if and only if $\underline{a} > 0$. The corresponding equivalence for $\overline{\det(\boldsymbol{A})} < 0$ can be derived in a similar way. Therefore, if we had such an ε-approximation algorithm, from the sign of the returned determinant enclosure the regularity can be decided. □

Theorem 8.9 (Absolute approximation, Horáček et al. [86])**.** *Let A_c be a rational nonnegative symmetric positive definite $n \times n$ matrix. Let $\boldsymbol{A} = [A_c - E, A_c + E]$ and let ε be arbitrary such that $0 < \varepsilon$. If there exists a polynomial time algorithm returning $[\underline{a}, \overline{a}]$ such that*

$$\det(\boldsymbol{A}) \subseteq [\underline{a}, \overline{a}] \subseteq \det(\boldsymbol{A}) + [-\varepsilon, \varepsilon],$$

then $\mathsf{P} = \mathsf{NP}$.

Proof. We again use the fact from Theorem 8.6 and show that if such an ε-approximation algorithm existed, then we can decide the coNP-complete problem. Which would imply $\mathsf{P} = \mathsf{NP}$.

Let the matrix A_c consist of rational numbers with nominator and denominator representable by k bits (we can take k as the maximum number of bits needed for any nominator or denominator). Then nominators and denominators of coefficients in $A_c - E$ and $A_c + E$ are also representable using $O(k)$ bits. Each row of the matrices is now multiplied with a product of all denominators from the corresponding row of both $A_c - E, A_c + E$. Each denominator still uses k bits and each nominator uses $O(nk)$ bits. We obtained a new matrix $\boldsymbol{A}'$. The whole matrix now uses $O(n^3k)$ bits which is polynomial in n and k.

We only multiplied by nonzero constants therefore the following property holds

$$0 \notin \det(\boldsymbol{A}) \iff 0 \notin \det(\boldsymbol{A}').$$

After canceling fractions, the matrix $\boldsymbol{A}'$ has integer bounds. Its determinant must also have integer bounds. Therefore deciding whether $\boldsymbol{A}'$ is regular means deciding whether $|\det(\boldsymbol{A}')| \geq 1$. We can multiply one arbitrary row of $\boldsymbol{A}'$ by 2ε and get a new matrix $\boldsymbol{A}''$ having $\det(\boldsymbol{A}'') = 2\varepsilon \det(\boldsymbol{A}')$. Now, we can apply the approximation algorithm and compute an absolute approximation $[\underline{a}'', \overline{a}'']$ of the determinant of $\boldsymbol{A}''$. Let $\det(\boldsymbol{A}') \geq 1$. Then $\det(\boldsymbol{A}'') \geq 2\varepsilon$ and the lower bound of the absolute approximation is

$$\underline{a}'' \geq \underline{\det(\boldsymbol{A}'')} - \varepsilon \geq \varepsilon > 0,$$

On the other hand, if $\underline{a}'' > 0$ then

$$2\varepsilon \, \underline{\det(\boldsymbol{A}')} = \underline{\det(\boldsymbol{A}'')} \geq \underline{a}'' > 0.$$

Hence, even $\underline{\det(\boldsymbol{A}')} > 0$ and since it is an integer it must be greater or equal to 1. The case of $\overline{\det(\boldsymbol{A}')} \leq -1$ is handled similarly. Therefore, we proved

$$0 \notin \det(\boldsymbol{A}) \iff 0 \notin \det(\boldsymbol{A}') \iff 0 \notin [\underline{a}'', \overline{a}''].$$

That means we can decide regularity with our ε-approximation algorithm. □

8.4 Enclosure of a determinant: general case

8.4.1 Gaussian elimination

To compute an enclosure of the determinant of an interval matrix Gaussian elimination introduced in Chapter 5 can be used – after transforming a matrix into row echelon form an enclosure of the determinant is computed as the product of the intervals on the main diagonal. We remind that, as in the real case, swapping of two rows changes the sign of the resulting enclosure.

It is usually favorable to use Gaussian elimination together with a preconditioning (more details will be explained in Subsection 8.4.6). We would recommend the midpoint inverse preconditioning as was done in [208].

Example 8.10. Because of properties of interval arithmetic (subdistributivity) interval Gaussian elimination leads to a certain overestimation. Let us have a matrix

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} \end{pmatrix}.$$

In Section 4.2 we computed the hull of the determinant of such a matrix as $\det(\boldsymbol{A}) = \boldsymbol{a}_{11} \cdot \boldsymbol{a}_{22} - \boldsymbol{a}_{12} \cdot \boldsymbol{a}_{21}$ (the determinant of a 2×2 matrix is a formula with single occurrence of each matrix coefficient and we can apply Theorem 3.13).

After one elimination step we get the matrix

$$\begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} \\ 0 & \boldsymbol{a}_{22} - \frac{\boldsymbol{a}_{21}}{\boldsymbol{a}_{11}} \cdot \boldsymbol{a}_{12} \end{pmatrix}.$$

The following holds according to subdistibutivity and nonexistence of inverse element in the interval arithmetics.

$$\boldsymbol{a}_{11} \cdot \left(\boldsymbol{a}_{22} - \frac{\boldsymbol{a}_{21}}{\boldsymbol{a}_{11}} \cdot \boldsymbol{a}_{12} \right) \supseteq \boldsymbol{a}_{11} \cdot \boldsymbol{a}_{22} - \boldsymbol{a}_{12} \cdot \boldsymbol{a}_{21} = \det(\boldsymbol{A}).$$

8.4.2 Gerschgorin discs

It is a well-known result that the determinant of a real matrix is a product of its eigenvalues. That is why an enclosure of an interval determinant can be computed as a product of enclosures of interval matrix eigenvalues, e.g., [69, 78, 108, 124]. The Gerschgorin circle theorem can be used as well.

This classical result claims that for a real square matrix A each its eigenvalue lies inside at least one *Gerschgorin disc* in complex plane with centers A_{ii} and radius $\sum_{j \neq i} |A_{ij}|$. When $\boldsymbol{A}$ is an interval matrix, to each real matrix $A \in \boldsymbol{A}$ there corresponds a set of Gerschgorin discs. Increasing or decreasing the coefficients of A within $\boldsymbol{A}$ shifts or scales the discs. However, all discs corresponding to ith diagonal element of $\boldsymbol{A}$ in all situations are contained inside a disc with the center $\mathrm{mid}(\boldsymbol{A}_{ii})$ and the radius $\mathrm{rad}(\boldsymbol{A}_{ii}) + \sum_{j \neq i} \mathrm{mag}(\boldsymbol{A}_{ij})$ as depicted in Figure 8.2. We can call such a disc an *interval Gerschgorin disc*.

As in the case of the real Gerschgorin discs, it is also well known that in the union of k intersecting discs there somewhere lie k eigenvalues. By intersecting discs we mean that their projection on the horizontal axis is a continuous line. That might complicate the situation a bit. When k interval Gerschgorin discs intersect each $A \in \boldsymbol{A}$ specifies a distribution of k eigenvalues in the bunch of the k interval discs.

That is why we can deal with each bunch of intersecting discs separately. We compute the verified interval enclosing all products of k eigenvalues regardless of their position inside this bunch. The computation of the verified enclosures will depend on the number of discs in the bunch (odd/even) and on whether the bunch contains the point 0. In Figures 8.3 and 8.4 all the possible cases and resulting verified enclosures are depicted. The resulting determinant will be a product of intervals corresponding to all bunches of intersecting discs.

The formulas for enclosures of a bunch of discs are based on the following simple fact depicted in Figure 8.5: an eigenvalue lying inside an intersection of two discs can be real or complex $(c + bi)$. In the second case the conjugate complex number $c - bi$ is also an eigenvalue. Their product is $b^2 + c^2$, which can be enclosed from above by a^2, where a is defined in Figure 8.5. The whole reasoning is based on Pythagorean theorem and geometric properties of hypotenuse.

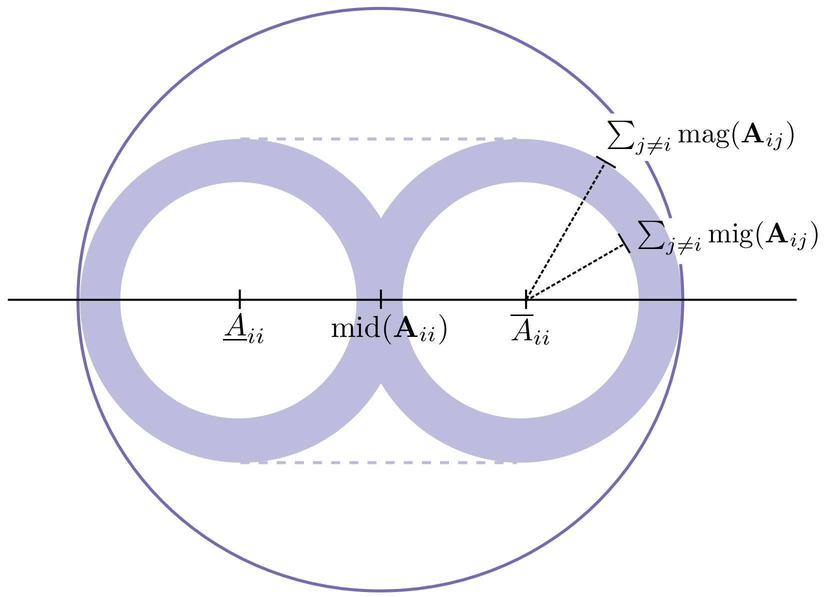


Figure 8.2: One interval Gerschgorin disc (the large circle). The grey area mirrors the scaling and shifting of a real Gerschgorin disc when shifting coefficients of A within intervals of $\boldsymbol{A}$

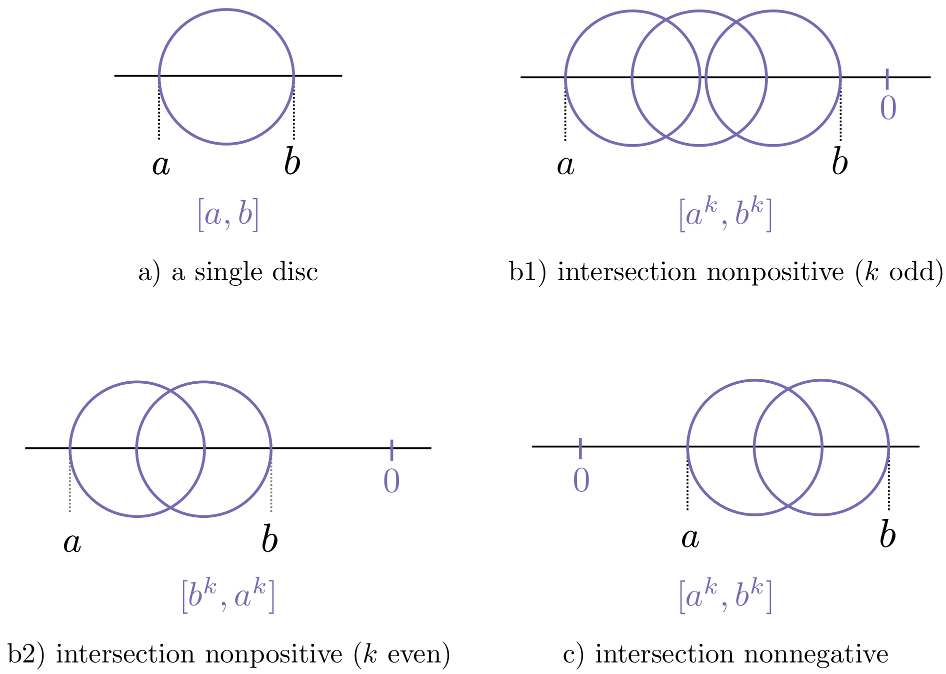


Figure 8.3: Verified enclosures of any product of real eigenvalues inside a bunch of intersecting interval Gerschgorin discs not containing 0.

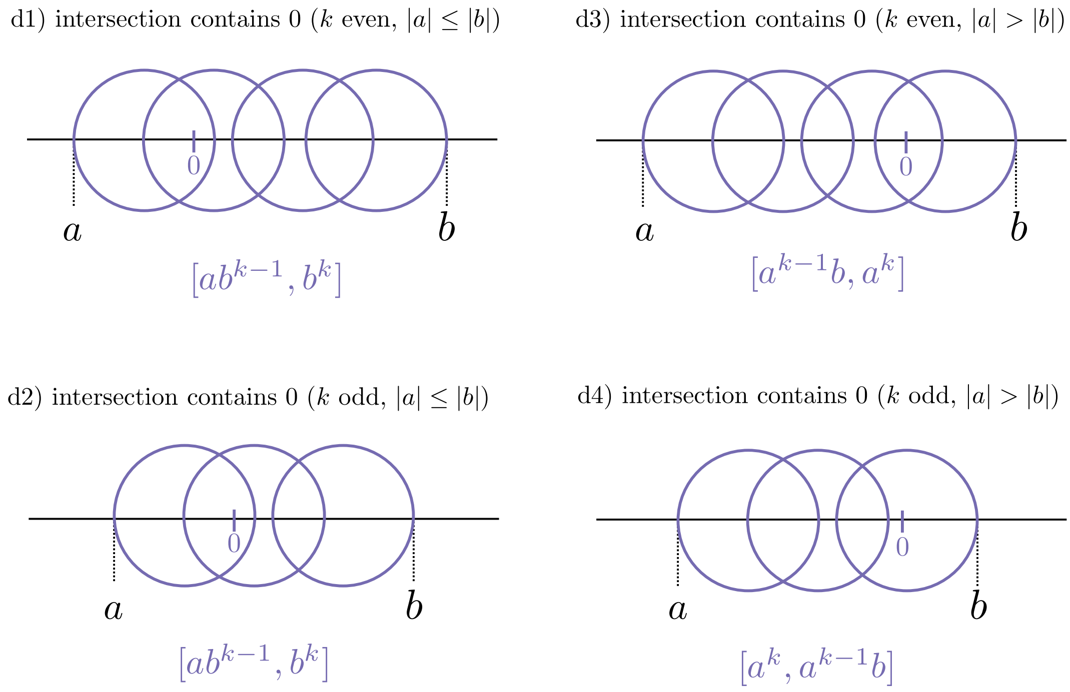


Figure 8.4: Verified enclosures of any product of real eigenvalues inside a bunch of intersecting interval Gerschgorin discs containing 0.

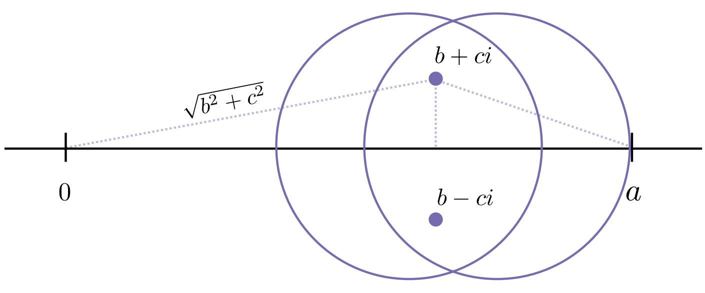


Figure 8.5: Enclosing a product of two complex eigenvalues.

The generalized interval Gerschgorin discs approach may produce large overestimation. However, it might be useful in case of tight intervals or a matrix close to a diagonal one.

8.4.3 Hadamard's inequality

A simple but rather crude enclosure of interval determinant can be obtained by the well known Hadamard's inequality. For an $n \times n$ real matrix A we have

$$|\det(A)| \leq \prod_{i=1}^{n} \|A_{*i}\|_2 = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |A_{ji}|^2 \right)^{\frac{1}{2}},$$

where $\|A_{*i}\|_2$ is the Euclidean norm of the ith column of A. This inequality is simply transferable to the interval case. Since the inequality holds for every $A \in \boldsymbol{A}$ we have

$$\det(\boldsymbol{A}) \subseteq [-d, +d]\,, \text{ where } d = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} \operatorname{mag}(\boldsymbol{A}_{ji})^2 \right)^{\frac{1}{2}}.$$

Since $\det(A) = \det(A^T)$, the same formula can be computed also for rows instead of columns and intersection of the two determinant enclosures can be taken. It is a fast and simple method. A drawback is that the obtained enclosure is often wide. A second problem is that it is impossible to detect the sign of the determinant.

8.4.4 Cramer's rule

In this section we introduce our method that is based on Cramer's rule [86]. In Chapter 5 we introduced various methods for computing an enclosure of the solution set of a square interval linear system and we can again make use of them. According to Cramer's rule for a real system of equations $Ax = b$ we get

$$x_1 = \frac{\det(A_{1\leftarrow b})}{\det(A)},$$

where x_1 is the first coefficient of the solution vector x and $A_{1\leftarrow b}$ is the matrix that emerges when we substitute the first column of A with b. We can rewrite the equation as

$$\det(A) = \frac{\det(A_{1\leftarrow b})}{x_1}.$$

Let $b = e_1$ and let us assume that we know x_1 from solving a system $Ax = b$ then $\det(A_{1\leftarrow b})$ is equal to $\det(A_{2:n})$ which emerges by omitting the first row and column from A. Now, we have reduced our problem to computing determinant of a matrix of lower order and we can repeat the same procedure iteratively until the determinant is easily computable. Such a procedure will not pay off in the case of real matrices. However, it will help in the interval case. We actually get

$$\det(\boldsymbol{A}) \subseteq \det(\boldsymbol{A}_{2:n})/\boldsymbol{x}_1, \tag{8.3}$$

where $\boldsymbol{x}_1$ is the interval enclosure of the first coefficient of the solution of $\boldsymbol{A}x = e_1$, computed by some of the cited methods. Notice that we can use arbitrary e_i instead of e_1. The method works when all enclosures of $\boldsymbol{x}_1$ in the recursive calls (8.3) do not contain 0.

8.4.5 Monotonicity checking

According to [149], the partial derivatives of $\det(A)$ of a real nonsingular matrix $A \in \mathbb{R}^{n\times n}$ are

$$\frac{\partial \det(A)}{\partial A} = \det(A)A^{-T}.$$

Let $\boldsymbol{B}$ be an an interval enclosure for the set $\{A^{-T} \mid A \in \boldsymbol{A}\}$. Since $\boldsymbol{A}$ is regular, every $A \in \boldsymbol{A}$ has the same sign of determinant. Hence, e.g., $\det(A_c)\boldsymbol{B}_{ij}$ gives information about monotonicity of the determinant.

When long as 0 is not in the interior of $\boldsymbol{B}_{ij}$, then we can do the following reasoning: if $\det(A_c)\boldsymbol{B}_{ij}$ is a nonnegative interval, then $\det(A)$ is nondecreasing in A_{ij}, and hence its minimal value is attained at $A_{ij} = \underline{A}_{ij}$. Similarly for $\det(A_c)\boldsymbol{B}_{ij}$ nonpositive.

In this way, we split the problem of computing $\det(\boldsymbol{A})$ into two subproblems of computing the lower and upper bounds separately. For each subproblem, we can fix those interval entries of $\boldsymbol{A}$ at the corresponding lower or upper bounds depending on the signs of $\boldsymbol{B}_{ij}$. This makes the set $\boldsymbol{A}$ smaller in general. We can repeat this process or call another method for the reduced interval matrix.

Notice that there are classes of interval matrices with monotone determinant. They are called inverse stable [169]. Formally, $\boldsymbol{A}$ is inverse stable if $|A^{-1}| > 0$ for each $A \in \boldsymbol{A}$. This class also includes interval M-matrices [12], inverse nonnegative [117] or totally positive matrices [45] as particular subclasses that are efficiently recognizable; cf. [75].

8.4.6 Preconditioning

In an interval case by preconditioning we mean transforming an interval matrix into a form that is more suitable for further processing. It is generally done by multiplying an interval matrix $\boldsymbol{A}$ with some real matrices B, C from the left and right respectively.

$$\boldsymbol{A} \quad \mapsto \quad B\boldsymbol{A}C.$$

Regarding interval determinant, we have the following result.

Proposition 8.11. *Let $\boldsymbol{A}$ be a square interval matrix and let B, C be real square matrices of the corresponding size. Then*

$$\det(B) \cdot \det(\boldsymbol{A}) \cdot \det(C) \subseteq \det(B\boldsymbol{A}C).$$

Proof. For any $A \in \boldsymbol{A}$ we have $\det(B) \cdot \det(A) \cdot \det(C) = \det(ABC) \in \det(B\boldsymbol{A}C)$. $\square$

We will further use the consequence

$$\det(\boldsymbol{A}) \subseteq \frac{1}{\det(B) \cdot \det(C)} \cdot \det(B\boldsymbol{A}C).$$

There are many possibilities how to choose the matrices B, C for a square interval matrix. First, we can use the approach from [208] – take the midpoint matrix A_c and compute its LU decomposition $PA_c = LU$, where L is a lower triangular matrix having ones on the main diagonal, U is upper triangular and P is a permutation matrix. Obviously, $\det(L) = \det(L^{-1}) = 1$. Determinant of P is 1 or -1. We take $B \approx L^{-1}$ (the main diagonal of B is set to ones) and $C = I$. Then according to Proposition 8.11 we have that

$$\det(\boldsymbol{A}) \subseteq \frac{1}{\det(P)} \cdot \det(L^{-1}P\boldsymbol{A}).$$

The resulting preconditioned interval matrix should be "close" to the upper triangular matrix U. We assume that such a preconditioning might be favorable for Gaussian elimination, since the preconditioned matrix is already close to row echelon form.

For a symmetric matrix an LDL^T decomposition can be used. A symmetric matrix A can be decomposed as $A = LDL^T$, where L is upper triangular with ones on the main diagonal and D is a diagonal matrix. Similarly, as in the previous case, we set $B \approx L^{-1}, C \approx L^{-T}$ and obtain

$$\det(\boldsymbol{A}) \subseteq \det(L^{-1}\boldsymbol{A}L^{-T}).$$

The resulting preconditioned interval matrix should be "close" to the diagonal matrix D.

For solving interval linear systems, there are various preconditioners used [74, 103]. The most common choice is taking $B = A_c^{-1}$ when A_c is nonsingular and $C = I$. Such a choice of B, C is also optimal in a certain sense [137, 139]. Of course, we are computing in a finite precision arithmetic, therefore we take only some approximation $B \approx A_c^{-1}$. According to Theorem 8.11 we get

$$\det(\boldsymbol{A}) \subseteq \det(A_c^{-1}\boldsymbol{A}) / \det(A_c^{-1}).$$

Notice that the matrix A_c^{-1} does not generally have its determinant equal to 1. That is why we need to compute a verified determinant of a real matrix. We present an example of such an algorithm in the next section.

8.5 Verified determinant of a real matrix

In [145] a variety of algorithms for computation of verified determinant of real matrices is presented. We are going to use the simplest one by Rump [195]. For a real square matrix X we compute its LU decomposition using the floating point arithmetics such that

$$PX \approx LU,$$

where L is lower triangular, U is upper triangular and P is a permutation matrix following partial pivoting (therefore $\det(P) = \pm 1$). Let X_L, X_U be approximate inverses of L, U respectively. We force X_L to be lower triangular with unit main diagonal (therefore $\det(X_L) = 1$). We denote $Y := X_L P X X_U$. We enclose the coefficients of X with verified intervals and obtain an interval matrix $\boldsymbol{X}$. Therefore, the resulting matrix $\boldsymbol{Y} = X_L P \boldsymbol{X} X_U$ will be close to the identity matrix and its determinant is close to 1. To compute its determinant, we can apply, e.g., the interval version of the Gerschgorin circle theorem (Section 8.4.2). From

$$\det(Y) = \det(P)\det(X)\det(X_U).$$

we get

$$\det(X) = \frac{1}{\det(P)} \cdot \frac{\det(Y)}{\det(X_U)}.$$

We can also enclose the diagonal elements of X_U with tight intervals and compute its determinant simply as a product of these intervals. If $0 \notin \det(\boldsymbol{X}_U)$ we get

$$\det(X) \in \det(\boldsymbol{X}) \subseteq \frac{1}{\det(P)} \cdot \frac{\det(\boldsymbol{Y})}{\det(\boldsymbol{X}_U)}.$$

8.6 Enclosure of a determinant: special cases

Even though we are not going to compare all of the mentioned methods in this section, for the sake of completeness, we will mention some cases of matrices, that enable the use of another tools. For some classes of interval matrices tasks connected to determinants are computable efficiently.

8.6.1 Symmetric matrices

Many problems in practical use are described by symmetric matrices. In connection with determinant a new approach can be used. We specify what we mean by an interval symmetric matrix in the following definition.

Definition 8.12 (Symmetric interval matrix)**.** For a square interval matrix $\boldsymbol{A}$ we define the symmetric matrix $\boldsymbol{A}^S$ as

$$\boldsymbol{A}^S = \{A \in \boldsymbol{A} \mid A = A^T\}.$$

Its eigenvalues are defined as follows.

Definition 8.13. For a real symmetric matrix A let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be its eigenvalues. For $\boldsymbol{A}^S$ we define its ith set of eigenvalues as $\boldsymbol{\lambda}_i(\boldsymbol{A}^S) = \{\lambda_i(A) \mid A \in \boldsymbol{A}^S\}$.

For symmetric interval matrices there exist various methods to enclose each ith set of eigenvalues. A proposition by Rohn [175] gives a simple enclosure.

Proposition 8.14. $\boldsymbol{\lambda}_i(\boldsymbol{A}^S) \subseteq [\lambda_i(A_c) - \varrho(A_\Delta), \lambda_i(A_c) + \varrho(A_\Delta)]$.

The previous proposition requires computation of verified enclosures of eigenvalues of real matrices; for more details on such an issue see, e.g., [128, 129, 221].

There exist various other approaches for computing enclosures of the eigenvalues (e.g., [107, 119]), there are several iterative improvement methods (e.g., [15, 79]). For the exact minimum and maximum extremal eigenvalues, there is a closed-form expression [64], which is however exponential.

8.6.2 Symmetric positive definite matrices

Let $\boldsymbol{A}^S$ be a symmetric (strongly) positive definite matrix, that is, every $A \in \boldsymbol{A}^S$ is positive definite. For more details about positive definite matrices see Section 11.10.

The matrix with maximum determinant can be found by solving the optimization problem

$$\max \ \log\det(A) \text{ subject to } A \in \boldsymbol{A}^S.$$

The condition $A \in \boldsymbol{A}^S$ can be rewritten as linear conditions

$$\forall i,j \quad \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, \quad \forall i \neq j \quad a_{ij} = a_{ji},$$

and the function $\log\det(A)$ is a so-called *self-concordant* function for which such an optimization problem is solvable in polynomial time with respect to dimension of a problem and $1/\varepsilon$ (where ε is a desired accuracy) using interior point methods; see Boyd and Vandenberghe [22]. Therefore, we have:

Proposition 8.15. *The maximum determinant of a symmetric positive definite matrix is computable in polynomial time.*

8.6.3 Matrices with $A_c = I$

Preconditioning $\boldsymbol{A}$ by A_c^{-1} results in an interval matrix with I as the midpoint matrix. We saw that such matrices imply favorable properties (polynomial hull computation – Subsection 5.6.2, nicer sufficient conditions for regularity – Section 4.1).

Proposition 8.16. *Suppose that $\varrho(A_\Delta) < 1$. Then the minimum determinant of $\boldsymbol{A}$ is attained for $\underline{A}$.*

Proof. According to Corollary 4.3 the fact $\varrho(A_\Delta) < 1$ implies regularity of $\boldsymbol{A}$; and also of $\underline{A}$.

We will proceed by mathematical induction. For $n = 1$ the proof is trivial. For a general case, we express the determinant of $A \in \boldsymbol{A}$ as in (8.3)

$$\det(A) = \det(A_{2:n})/x_1. \tag{8.4}$$

Notice that $\boldsymbol{A}$ and $\boldsymbol{A}_{2:n}$ have identity matrices as midpoints, whose determinant is equal to 1. Regularity of every $A \in \boldsymbol{A}$, and hence of $A_{2:n} \in \boldsymbol{A}_{2:n}$, then implies

$$\det(A) > 0, \quad \det(A_{2:n}) > 0.$$

Therefore, we know that also $x_1 > 0$. To obtain lower bound on $\det(\boldsymbol{A})$ we need to minimize the numerator and maximize the denominator of (8.4). By induction hypothesis, the smallest value of $\det(A_{2:n})$ is attained for $A_{2:n} = \underline{A}_{2:n}$. The solution x of $Ax = e_1$ is the first column of A^{-1}. From Theorem 11.21 it follows that the upper bound on $\boldsymbol{A}^{-1}_{*1}$ is obtained by setting $A = (I - A_\Delta) = \underline{A}$. Therefore $A = \underline{A}$ simultaneously minimizes the numerator and maximizes the denominator of (8.4). □

Example 8.17. If the condition $\varrho(A_\Delta) < 1$ does not hold, then the claim is generally wrong. Let us have the matrix $\boldsymbol{A} = [A_c - A_\Delta, A_c + A_\Delta]$ where

$$A_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\Delta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

we have $\varrho(A_\Delta) = 3$ and $\det(\underline{A}) = -2$, however, the $\det(\boldsymbol{A}) = [-6, 14]$. The minimum bound is attained, e.g., for the matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The reasoning from the proof of Theorem 8.16 cannot be applied for computing the upper bound of $\det(\boldsymbol{A})$.

Example 8.18. For the matrix $\boldsymbol{A} = [A_c - A_\Delta, A_c + A_\Delta]$ where

$$A_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\Delta = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

we have $\varrho(A_\Delta) = 0.75 < 1$ and $\det(\boldsymbol{A}) = [0.25, 2.1875]$. However, $\det(\overline{A}) = 1.75$.

Computing the maximum determinant of $\boldsymbol{A}$ is a more challenging problem. It is an open question whether is can be done in polynomial time. Obviously, the maximum determinant of $\boldsymbol{A}$ is attained for a matrix $A \in \boldsymbol{A}$ such that $A_{ii} = \overline{A}_{ii}$ for each i. Specifying the off-diagonal entries is, however, not so easy.

8.6.4 Tridiagonal H-matrices

Consider an interval tridiagonal matrix

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{b}_2 & 0 & \dots & 0 \\ \boldsymbol{c}_2 & \boldsymbol{a}_2 & \boldsymbol{b}_3 & \ddots & \vdots \\ 0 & \boldsymbol{c}_3 & \boldsymbol{a}_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{b}_n \\ 0 & \dots & 0 & \boldsymbol{c}_n & \boldsymbol{a}_n \end{pmatrix}.$$

Suppose that it is an interval H-matrix, which means that each matrix $A \in \boldsymbol{A}$ is an H-matrix (for a definition see Section 4.4). Without loss of generality let us assume that the main diagonal is positive, that is, $\underline{a}_i > 0$ for all $i = 1, \ldots, n$. Otherwise, we can multiply the corresponding rows by -1.

Recall that the determinant D_n of such a real tridiagonal matrix of order n can be computed by the recursive formula

$$D_n = a_n D_{n-1} - b_n c_n D_{n-2}.$$

Since $\boldsymbol{A}$ is an H-matrix with positive diagonal, the values of $D_1, \ldots, D_n$ are positive for each $A \in \boldsymbol{A}$ (see, e.g., [19]). Hence the largest value of $\det(A)$ is attained at $a_i := \overline{a}_i$ and b_i, c_i such that $b_i c_i = \underline{\boldsymbol{b}_i \boldsymbol{c}_i}$. Analogously for the minimal value of $\det(A)$. Hence we constructively proved the following proposition.

Proposition 8.19. *Determinants of interval tridiagonal H-matrices are computable in polynomial time.*

Complexity of determinant computation for general tridiagonal matrices remains an open problem, similarly as solving an interval system with tridiagonal matrix [112]. Nevertheless, not all problems regarding tridiagonal matrices are open or hard, e.g., deciding whether a tridiagonal matrix is regular can be done in polynomial time [11].

8.7 Comparison of methods

In this section some of the previously described methods are compared. First, we start with general square matrices. Then we test on symmetric matrices. All the tests were computed using the DESKTOP setup (see Section 3.11).

8.7.1 General case

For general matrices the following methods are compared:

- `ge` - interval Gaussian elimination,
- `cram` - our method based on Cramer's rule with HBR method for solving square interval systems,
- `had` - interval Hadamard's inequality,
- `gersch` - interval Gerschgorin circles.

The suffix `+inv` is added when the preconditioning with midpoint inverse was applied and the suffix `+lu` is added when the preconditioning based on LU decomposition was used. We use the label `hull` to denote the exact interval determinant.