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Interval linear and nonlinear systems

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Title: Interval linear and nonlinear systems

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Abstract: First, basic aspects of interval analysis, roles of intervals and their applications are addressed. Then, various classes of interval matrices are described and their relations are depicted. This material forms a prelude to the unifying theme of the rest of the work – solving interval linear systems.

Several methods for enclosing the solution set of square and overdetermined interval linear systems are covered and compared. For square systems the new shaving method is introduced, for overdetermined systems the new subsquares approach is introduced. Detecting unsolvability and solvability of such systems is discussed and several polynomial conditions are compared. Two strongest conditions are proved to be equivalent under certain assumption. Solving of interval linear systems is used to approach other problems in the rest of the work.

Computing enclosures of determinants of interval matrices is addressed. NP-hardness of both relative and absolute approximation is proved. New method based on solving square interval linear systems and Cramer's rule is designed. Various classes of matrices with polynomially computable bounds on determinant are characterized. Solving of interval linear systems is also used to compute the least squares linear and nonlinear interval regression. It is then applied to real medical pulmonary testing data producing several potentially clinically significant hypotheses. A part of the application is a description of the new breath detection algorithm. Regarding nonlinear systems an approach to linearizing a constraint satisfaction on an interval box problem into a system of real inequalities is shown. Such an approach is a generalization of the previous work by Araya, Trombettoni and Neveu. The features of this approach are discussed.

At the end computational complexity of selected interval problems is addressed and their feasible subclasses are captured. The interval toolbox LIME for Octave and its interval package, which implements most of the tested methods, is introduced.

Keywords: interval matrix, interval linear system, interval linear algebra, constraint satisfaction problem, interval regression, interval determinant, computational complexity

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Contents

1 Introduction

"To develop a complete mind: Study the art of science; study the science of art. Learn how to see. Realize that everything connects to everything else."

A quote attributed to Leonardo DaVinci.

In applications, many problems can be transformed to solving a system of linear equations. That is why linear systems often play a prominent role. There are various reasons why to incorporate intervals into such systems (rounding errors, inaccuracy of measurement, uncertainty, etc.). Similarly, in this work the main theme that weaves through all the chapters are interval linear systems. Of course, during the work we also meet nonlinear problems. However, it will be possible to deal with them using the linear means.

The first goal of this work is to present our contributions to several areas of interval analysis. Moreover, the work is submitted as a doctoral thesis of the author.

Most chapters are based on reworked and extended journal or conference papers published with other co-authors; mainly Michal Černý, Milan Hladík, Jan Horáček and Václav Koucký. Some of the results were also a product of joint work with defended students that were supervised by the author of this work; namely Josef Matějka and Petra Pelikánová. Some chapters contain also unpublished results and new material.

Parts of the text keep a survey book style with links to other works to enable the reader (either a professional or a student) to quickly pick up basics of the addressed area. That is the third goal of this work.

The material of this work is built in a cumulative way, hence a chapter usually uses the material from the previous chapters. However, we believe that each chapter could be read in a stand-alone manner with only occasional turning of pages. Most of the chapters are concluded with references to broader literature on various topics.

The work is divided into 12 chapters. Below is the brief content of each chapter:

Chapter 2. Roles of intervals. We introduce our understanding of intervals and their roles • We show simple examples comprehensible without knowledge of interval analysis • We discuss properties and advantages of intervals • The literature concerning applications and various ares of interval analysis referenced.

Chapter 3. Basic notation and ideas. Basic noninterval notation is introduced • Interval notation, concepts and structures we use are introduced • We briefly discuss the relation between intervals and rounded arithmetics • We discuss testing of interval methods and how to compare interval results.

Chapter 4. Interval matrix ZOO. Known classes of interval matrices, their properties and examples are presented • Their relations are depicted.

Chapter 5. Square interval linear systems. Various known direct and iterative methods are discussed • We discuss related topics such as preconditioning, finding initial enclosures and stopping criteria • We introduce the shaving method that enables further improvement of an enclosure • The methods are briefly compared.

Chapter 6. Overdetermined interval linear systems. The least squares solution is discussed • Various known methods for solving square interval linear systems are adapted for solving overdetermined interval systems • We introduce some known methods for solving overdetermined systems. • We introduce the subsquares approach and its variants.

Chapter 7. (Un)solvability of interval linear systems. Various conditions for detecting unsolvability are introduced • Checking full column rank is discussed and two sufficient conditions are proved to be equivalent under certain assumption • Checking solvability is addressed • The mentioned methods are compared.

Chapter 8. Determinant of an interval matrix. Known results about interval determinants are addressed • NP-hardness of absolute and relative approximation is proved • Known methods are refined to compute determinants of interval matrices • A method based on Cramer's rule is designed • Determinants of symmetric matrices are addressed • Classes of matrices with polynomially computable tasks related to interval determinants are explored • The methods are tested.

Chapter 9. Application of intervals to medical data. The Multiple breath washout procedure for lung function testing is introduced • Our algorithm for finding breath ends is introduced • Special type of regression where matrix is integer is discussed • Interval regression is applied to clinical data • Hypothetical conclusions are derived from the results.

Chapter 10. A linear approach to CSP. Linearization of nonlinear constraints is discussed • Linear programming approach is introduced • Vertex selection for linearization is discussed • Nonvertex selection for linearization is discussed • Properties of the proposed linearization are analyzed.

Chapter 11. Complexity of selected interval problems. Computational complexity in relation to intervals is explained • Complexity of various problems is addressed • Polynomially computable cases or classes of problems are characterized.

Chapter 12. LIME^2: interval toolbox. Interval toolbox LIME is introduced • Properties and goals of LIME are specified • Features and methods of LIME are listed • Installation and use is described.

1.1 Main results of the work

Here, we briefly summarize the main results of the work:

- **Chapter** 4. In this chapter we restructure results by Neumaier and others. New examples are added and relations between classes of interval matrices are analyzed and clearly visualized.

- **Chapter** 5. Many methods for solving interval linear systems need to use preconditioning. However, such an operation typically enlarges the original solution set. Methods applied to a preconditioned system return an enclosure of the enlarged solution set. In such cases a method that can further improve such an enclosure is of high importance. We introduce the shaving method that takes an enclosure and iteratively tries to shave off slices of the enclosure to get closer to the original solution set.

- **Chapter** 6. We shed more light on known methods for solving overdetermined interval linear systems. For overdetermined systems of interval linear equations we designed a new subsquares approach and its variants that can be easily parallelized and most of all can detect unsolvability of the system.

- **Chapter** 7. We describe several conditions for checking unsolvability and solvability of interval linear systems. Two conditions for detecting unsolvability that are based on full column rank detection are proved to be equivalent under certain assumption. Range of application of all conditions is visualized using heat maps.

- **Chapter** 8. In this chapter we prove that computing both the relative and absolute approximation of the exact determinant of an interval matrix is NP-hard. We characterize several classes of matrices with polynomially computable bounds on interval determinant. We design a new faster algorithm, based on Cramer's rule, for computing enclosure of the determinant of an interval matrix.

- **Chapter** 9. The interval least squares regression is applied to real world medical data from lung function testing. We show how to improve computation speed for certain regression input. We developed a new algorithm for detecting breath ends in clinical data. Such an algorithm can outperform the state-of-the-art algorithms even a commercial one. Based on the results we derive several hypotheses. If they turn to be true, it would have a significant impact on the area of current lung assessment methods.

- **Chapter** 10. Nonlinear constraint satisfaction problems are part of many practical problems. In this chapter we show how to linearize the nonlinear constraints and solve them using linear programming. For such a linearization an expansion point is needed. Older approaches used vertex points of the initial box, we show how to use an arbitrary point from the box. We prove that such a linearization is never worse then Jaulin's bounding with two parallel affine functions.

- **Chapter** 11. We discuss the complexity issues related to interval linear algebra. Then we provide a concise survey of complexity of selected problems from interval linear algebra.
- **Chapter** 12. We briefly introduce LIME, our interval package for Octave. Such package contains most of the methods mentioned in this work and some more.

For the list of author's publications and defended students see Chapter 13.

2 Roles of intervals

- Various roles of intervals
- Early use of intervals
- Properties and advantages of intervals
- Literature and sources on intervals

In this chapter various roles of intervals are demonstrated. They are introduced via examples that do not require a proper definition of an interval arithmetics yet. Later, useful properties and advantages of intervals are pointed out. We slightly mention the early works concerning intervals. The chapter is concluded with references to applications and other aspects of intervals.

2.1 Examples of intervals

Let us start with six simple examples. They illustrate various roles intervals can play.

Example 2.1. One of the earliest works on intervals was probably by Archimedes (287–212 BC). In his treatise Measurement of a Circle he gave the following verified bounds for π

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

Note that he proved that π indeed lies in the given bounds.

Example 2.2. Let us say, we want to know what time t it will take an object (simulated as a mass point) to fall from height $h = 50$ meters. If we take the h as a constant, then time t can be simply expressed as a function of a gravitational acceleration g as

$$t(g) = \sqrt{\frac{2h}{g}} = \frac{10}{\sqrt{g}} \text{ seconds.}$$

However, gravitational acceleration differs at various places on Earth as elaborated in the work [66]. The lowest estimated value $\underline{g}$ is on the Nevado Huascarán summit, Peru and the highest value $\overline{g}$ is on the surface of the Arctic Ocean

$$\underline{g} = 9.76392\ ms^{-2} \quad \overline{g} = 9.83366\ ms^{-2}.$$

If we do not know the exact g of our area we should simultaneously evaluate the formula for all g's of all measured surface points. However, since the function t is decreasing in g it is enough to evaluate it for $\overline{g}$ to obtain the shortest time and for $\underline{g}$ to obtain the longest time. Hence, when not computing with a specific g the time lies in the interval

$$[t(\overline{g}), t(\underline{g})] = [3.1889\ldots, 3.2002\ldots].$$

To make the bounds to safely contain the value of the time t we can say

$$t \in [3.1889, 3.2003] \text{ seconds.}$$

Example 2.3. Let us take the continuous function

$$f(x) = x^3 - 10x^2 + 27x - 18,$$

and let us inspect the existence of a root on the interval $[2, 5]$. The function f is continuous on $[2, 5]$, hence the intermediate value theorem states that f takes any value between $f(2)$ and $f(5)$ on this interval. As $f(2) = 4$ and $f(5) = -8$ the function f must take zero for some point in $[2, 5]$. Therefore, $[2, 5]$ is a verified interval containing a zero of the function f.

We can go further and use bisection – splitting the initial interval into halves and applying the intermediate value theorem on the two halves separately. If the function values at the endpoints of one half do not have different signs, then we go on to inspect the other half. The procedure can be recursively repeated. Here is the list of examined intervals safely containing a root.

$$\begin{aligned}
&[2, 5]\\
&[2, 3.5]\\
&[2.75, 3.5]\\
&[2.75, 3.125]\\
&[2.9375, 3.125]\\
&[2.9375, 3.03125]
\end{aligned}$$

If we properly handle the rounding errors the intervals introduce verified bounds on the location where the root lies. With each step the width of a resulting enclosure decreases. Since $f(x) = (x-1)(x-3)(x-6)$ we know that the exact root is 3. Such a method is very simple and can be further improved.

Example 2.4. A patient is connected to a breathing mask and instructed to breathe normally. During the breathing session various physical characteristics are measured by sensors in the mask. One of the variables measured is actual flow of air inside the mask. The sensor measures the flow value every given time slice. Let the length of a time-slice be d (usually $d = 5$ms). Moreover, the flow sensor has accuracy 5%. Hence, each measured flow in each time slice t denoted as φ_t becomes an interval

$$[0.95 \cdot \varphi_t,\ 1.05 \cdot \varphi_t].$$

Instead of a sequence of real numbers we get a sequence of intervals as depicted in Figure 2.1.

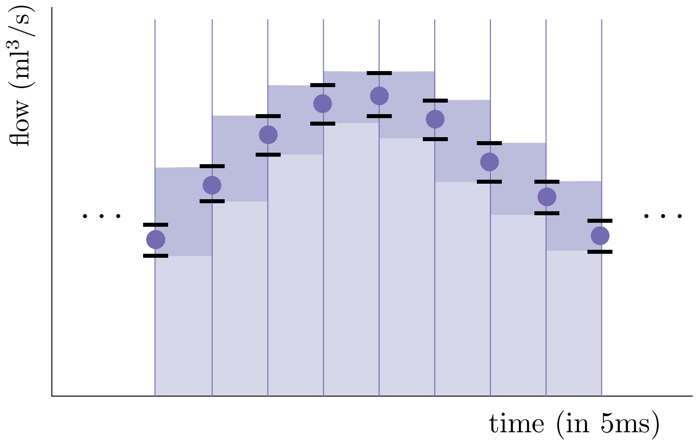

Figure 2.1: Simple verified volume computation. The circles represent flow measured in each time slice (vertical bars), short horizontal bars depict upper and lower bound on each measured flow incorporating 5 % measurement accuracy. Darker and lighter area depict the upper and lower bound on the volume respectively.

Many approaches to clinical assessment of lung function require the knowledge of the total volume of inhaled/exhaled air. Volume can be obtained as integration of flow – computing the area of the surface under the flow data. Since the time slice is small enough, the bounds for the volume can be computed as

$$\left[d \cdot \sum_{i=1}^{n-1} 0.95 \cdot \min(\varphi_i, \varphi_{i+1}),\ d \cdot \sum_{i=1}^{n-1} 1.05 \cdot \max(\varphi_i, \varphi_{i+1}) \right].$$

Such an approach can be used for other integration applications. The example is based on the real medical background later explained in Chapter 9. It is a philosophical question whether these bounds are verified (whether the measurement accuracy covers all phenomena that can occur). However, it is a safe way of using the measured data.

Example 2.5. Let us take the function f from the previous example. We want to inspect whether it is increasing on the interval $[5, 5.9]$. Since the first derivative of $f(x)$ is

$$f'(x) = 3x^2 - 20x + 27,$$

which is greater than 0 on $[5, 5.9]$. Thus f is increasing on this interval.

Example 2.6. Let us have one nonlinear constraint

$$x^2 - \cos(y) = 0,$$

where $x \in [-1, 1]$ and $y \in [-1, 1]$. Let us bound the feasible solutions of the constraint. The initial bounds on x and y can be further reduced.

For $y \in [-1, 1]$ the range of the function cos is included in $[0.54, 1]$. The maximum value is $\cos(0) = 1$ and the minimum value is $\cos(-1) = \cos(1) = 0.540302 \cdots > 0.54$.

Now, by expressing x as $|x| = \sqrt{\cos(y)}$ for $\cos(y) \in [0.54, 1]$ we get from monotonicity of $\sqrt{\cdot}$ that $|x| \in [0.73, 1]$, i.e,

$$x \in [-1, -0.73] \cup [0.73, 1].$$

Note that we actually proved that no solution has x, in the interval $[-0.7, 0.7]$,

In the above mentioned examples an interval played the following four roles:

1. interval in which a phenomenon occurs everywhere (Example 2.5),
2. interval in which a phenomenon occurs for sure, but we cannot tell where exactly (Example 2.1 and 2.3),
3. interval in which a phenomenon might occur (Example 2.2, 2.4 and 2.6),
4. interval in which a phenomenon does not occur (Example 2.6).

Such a perception of intervals is nothing new, we as people do it every day. The 1. is used when speaking of interval training (a form of training requiring to keep doing an exercise for a given period of time), a training when during an interval one must keep doing a prescribed activity followed by a short break. We use the 2. when watching Perseids or eclipse of the sun in the sky (these phenomenons have a known interval in which they occur). We use the 3. when placing a bet on a goal during a given period of a game. The 4. is used when referring to an amount of time between meals, a gap between objects, a break between two halves of a match.

In this work we are going to exploit these roles of intervals in various ways.

2.2 Application of intervals

Except from using the intervals in the way explained in the first section. The intervals can be, more specifically, used for various purposes:

- **To handle rounding errors.** By proper outward rounding of intermediate calculation results a verified interval containing the proper desired values can be obtained.
- **To express uncertainty.** In some situations we are not sure about the proper distribution of a phenomenon. Note that the situation is a bit different from having a uniform distribution on the interval. By uniform distribution we model the situation on an interval, however, in reality the obtained value can come from outside of the interval. Nevertheless, in the case of intervals the lower and upper bounds are verified to keep the value in between.
- **To cover measurement errors.** Machines have usually given operating accuracy in a form of ±error which produces interval bounds.
- **To proof a property for all representatives.** For example, in a dynamical system it is possible to prove that all points starting from a given initial area will reach an equilibrium.

Intervals can be used everywhere where the problems evince the kind of uncertainty already described – computer assisted proofs, economics, medicine, solving numerical systems and differential equations, constraint satisfaction problems and global optimization, computing physical constants, robotics, etc. It would be redundant to list all the possible applications, since it has been done many times. More uses of intervals can be found in [104, 132]. For more applications see, e.g., [5], [104]. The applications of intervals lies on many foundations.

2.3 Early works on intervals

We have already mentioned Archimedes and his approach to enclosing π with intervals. If we fast-forward to 20th century we encounter the following names in relation to intervals:

- **1931** – Rosalind Cecily Young published her paper *The algebra of multi-valued quantities* [222].
- **1951** – Paul Sumner Dweyer in his book *Linear computations* discusses range numbers and their use to measure rounding errors [36].
- **1956** – Mieczyslaw Warmus in his paper *Calculus of approximations* builds an interval apparatus for formulation of numerical problems [219].
- **1958** – Teruo Sunaga in his paper *Theory of interval algebra and its application to numerical analysis* develops interval calculus and shows its properties and examples in order to solve problems [209].
- **1961** – Ramon E. Moore published his Ph.D. thesis *Interval arithmetic and automatic error analysis in digital computing* [130].

This list is just to give a reader a brief peek into the historical connections of interval analysis. We are aware that this list is possibly very incomplete. History related to interval arithmetics is an interesting subject and would need much more space than we can afford here. More information regarding history of intervals can be found in [5, 196].[1]

Although the intervals were known early in 20th century it took some time before they were used in computers. There were possibly two reasons: interval operations were considered too slow in comparison with their real counterparts and the resulting intervals were huge. However, this comparison with real numbers was a bit unfair because interval computations solve a different problem – instead of "some" solution of unknown quality interval arithmetics gives us rigorous bounds for the solution. Regarding the widths of intervals, with the successive developments of new methods the resulting intervals have started to be of applicable quality.

[1] Many early papers on intervals are accessible at `http://www.cs.utep.edu/interval-comp/early.html` (Accessed February 10, 2019).

2.4 More on intervals

There are a lot of works to start with for better knowledge of intervals. A very short introduction is, e.g, [215] by Tucker or [104] by Kearfott. A classical book on introduction to interval analysis is [133] by Moore, Kearfott and Cloud. Another Moore's book on more mathematical applications of interval analysis is [132]. It contains large list of interval-related publications. Many key concepts are shown in another classical books – [3] by Alefeld and Herzberger and [139] by Neumaier. A book with applications mostly in robotics and control is [99] by Jaulin and et al. There is a work on verified numerics by Rump [193]. Regarding global optimization there is a book [59] by Hansen and Walster. A list of interval related publications is [51, 52]. All problems can be viewed from the computational complexity point of view. There is a thorough book [111] or one can read our survey paper on computational complexity and interval linear algebra [85]. Also Rohn's hanbook [176] can serve as a useful signpost to other interval topics. For introduction to computer (interval) arithmetic see, e.g., [115] or the IEEE interval standard [162].