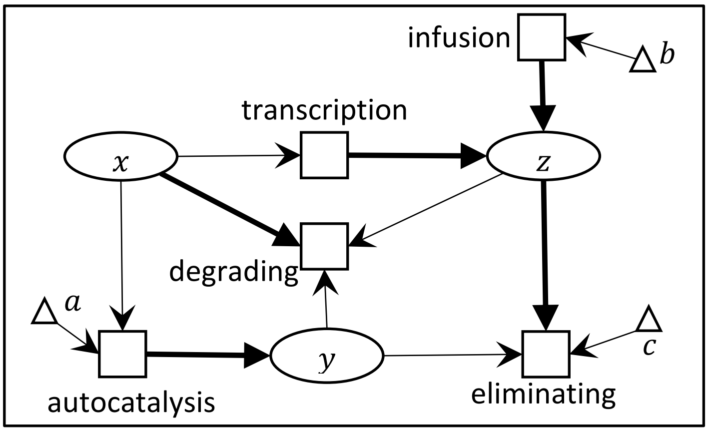
209. Evaluating systems for chaotic behaviour

What new skills will I possess after completing this laboratory?

- **Applying** criteria for recognising the transition of dynamical systems into chaos;
- **Evaluating** the dynamics of physical systems with respect to chaos.

Why do I need these skills?

In 1976, the PhD student Otto Rössler took on the task of designing a *chaotic* chemical reaction – that is, a chemical reaction whose dynamics we cannot predict in advance. He did not succeed in this task, but his attempts to solve the task led him to develop a particularly simple set of equations that display chaotic dynamics:



$$(1)\quad \begin{cases} \dot{x} = -(y+z) \\ \dot{y} = x + ay \\ \dot{z} = b + xz - cy \end{cases}$$

where a, b and c are all constants.

(i) Recall that the Poincaré-Bendixson theorem states that in order to display chaotic behaviour, a dynamical system must be both non-linear and possess at least three degrees of freedom. Rössler's equations are therefore only marginally capable of chaos: they form a third-order system, but violate linearity only in one place – where?

(ii) Run the method **Roessler.demo()** model over 200 seconds with $a = b = 0.2$, $c = 2.5$ and $x(0) = y(0) = z(0) = 1$. What archetypal behaviour do you observe in the BOTG?

(iii) Now set $c = 3.5$ and then $c = 4$. Do you recognise what is happening here?

(iv) Let's check this out. First, are we sure that period-doubling is occurring here? Check this by creating xy-phase-plots of the above experiments.

(v) These results are difficult to interpret due to the initial *transient* behaviour. Create a 300 s trace, then discard the transient first 100 seconds of this trace, to obtain more clearly analysable results for the parameter values $c \in \{2.5,3.5,4\}$.

What is the structure of the skills?

We have established the occurrence of period-doubling in the Rössler system. Period-doubling is a good indicator of the transition into chaos, so this information already suggests that Rössler had found something interesting. But we have a problem:

(vi) Trajectories in a phase portrait cannot cross themselves, because at such a crossover point, the system could not know in which direction to go along the trajectory. Yet in the phase diagram for $c = 3.5$, there appears to be a crossover! How can this happen?

(vii) Use the **Axis3()** constructor to view the behaviour of x, y and z for $c \in \{2.5,3.5,4,5\}$. Again, discard the first 100 s of a 300 s trace.

(viii) So: why do you think chaotic behaviour requires three dimensions in order for it to occur?

(ix) Now let's explore. Try out: $b = 0.2$, $c = 5.7$, $a \in \{-1.0, 0.0, 0.1, 0.2, 0.3, 0.35, 0.38, 4.0\}$.

(x) Now try: $a = 0.2$, $c = 5.7$, $b \in \{2.0, 1.0, 0.8, 0.73, 0.6, 0.5, 0.35, 0.3, 0.2, 0.1, 0.05, 0.01\}$.

How can I extend my skills?

A ***complex*** system is a system that exhibits complex behaviour. ***Complex behaviour*** is behaviour which is globally stable, but is generated by components that would generate chaotic behaviour if they were isolated from the rest of the system. Complex behaviour is an essentially systemic phenomenon that arises through ***self-organisation***: system-level influence adjusts the critical parameters of its components to regions where their behaviour is relative stable.

Self-organisation is a fundamental property of living systems. A living system must respond *flexibly* to its environment (think of a hare weaving to escape a lynx), and *chaotic* attractors provide this flexibility to react spontaneously. Yet living systems must also respond *robustly* to situations with predictable outcomes (think of the lynx weaving to catch the hare), and *simple* attractors provide this robustness. What links the two is *learning*, which enables the organism to decide in each new situation whether it should behave flexibly or robustly.

We will not discuss learning in this subject – that will have to wait until a later subject, when we look at agent-based systems. However, you have seen in exercise (x) just how unpredictable the jump between order and chaos can be. The following exercises in this unit suggest several different ways in which you can investigate and evaluate chaotic behaviour in a system such as the Rössler system.

(xi) **Estimation:** Look at your answer to exercise (x), and sketch by hand a map of where you think the stable and chaotic regions of the parameter b are.

(xii) **Poincaré section:** This tool is particularly useful for systems with many dimensions: the idea is that we reduce a higher-order system to an easily visualisable 2-dimensional plot by plotting only points where the trajectory crosses some appropriate phase-plane in a particular direction. On the website for **DynamicalSystems.jl**, look up the method **PoincareMap()**, then use this to analyse exercise (x) by plotting the xy-coordinates of points crossing the plane $z = z_0$ for some appropriate value of z_0.

(xiii) **Attractors:** Remember that any system will spend most of its time around an attractor. When we use a simulation to plot some interesting trajectory, that trajectory is only one of infinitely many such trajectories. We must always remember to inspect the trajectory carefully to see if it gives us clues about the location and type of attractors. Always think: *What this trajectory telling me – what deeper truth lies behind it?*

(xiv) **Automation:** It is a lot of hard work to set up a large number of trajectory experiments by hand. Remember to use the benefits of julia as a computer language for automating the performance of many experiments. Create a function **Roesslers.bifurc()** that generates a Poincaré section of the Rössler model for each of the values of b used in exercises (x) and (xii).

How can I deepen my practice of the skills?

(xv) In writing a report, it is extremely important to interpret your findings carefully. Nobody will be interested in your analysis unless you can interpret it for them, but your interpretation must also be accurate, or you will mislead your readers. In the SPD above, I have tried to interpret the Rössler model as a chemical system. Practise your interpretation skills now by constructing an interpretation of the Rössler system as a mechanical oscillator.