208. Bifurcation: How systems fail!

What new skills will I possess after completing this laboratory?

- **Applying and analysing** bifurcation diagrams
- **Analysing** the effect of critical parameters on phase portraits
- **Developing** behavioural predictions from bifurcation diagrams

Why do I need these skills?

So far, we have used the term *bifurcation* several times, but have not explained exactly what it is. In the Holling-Tanner model it was an abrupt change from point attractor to cycles as we varied the value of the specific prey consumption rate N; in the Griffith switch it was the appearance of two critical points as we reduced the degradation rate α of the protein.

A ***bifurcation*** is any value μ_0 of some system parameter μ, at which the system behaviour becomes *unstable*, that is, the system has a different kind or number of critical points for values of μ above and below μ_0 in the neighbourhood $[\mu_0 - \Delta\mu, \mu_0 + \Delta\mu]$ of μ_0.

Example: How does the behaviour of the system $\dot{x} = \mu - x^2$; $\dot{y} = -y$ depend on μ?

Solution: We can find the critical points of these equations by solving the equations $\dot{x} = \dot{y} = 0$, and the number of these solutions depends on the value of μ. For example, if $\mu < 0$, these equations have no real solutions and so there are no critical points. On the other hand, if $\mu = 0$, we find a single saddle node at $(0,0)$, and if $\mu > 0$, we find two nodes at $(\pm\sqrt{\mu}, 0)$.

(i) Run the method **Bifurcations.demo()** to see how this bifurcation develops as the parameter μ moves from positive into negative values.

(ii) The model of the above example shows how dangerous this bifurcation can be. Suppose the world is initially located at the positive stable node, so everything stays stable and seems to be ok. Now suppose that μ is not a constant parameter, but instead some physical property of our system's environment which drifts slowly over time from 1 down to -1. Watch this simulation again to see how the world's stability collapses as μ passes through zero!

This example is a ***saddle-node*** bifurcation. It has the canonical form $\dot{x} = \mu - x^2$. We sometimes call this a ***blue-sky*** bifurcation, because as μ increases from negative values, a neutral point emerges from nowhere at $x = \mu = 0$, splitting into *two* nodes as $\mu > 0$: *stable* at $x = +\sqrt{\mu}$ and *unstable* at $x = -\sqrt{\mu}$.

(iii) If you do not quite understand the danger of bifurcations yet, think about standing up in a boat. If you lean out just a little, everything stays comfortably stable, but if you lean just tiny a bit too far, you get another name for a bifurcation: a ***catastrophe***!

What is the structure of the skills?

There are many other kinds of bifurcation – let's explore some that are important in biology.

A ***transcritical*** bifurcation has the canonical form $\dot{x} = \mu x - x^2$. When $\mu < 0$, there are two critical points: an *unstable* node at $x = \mu$ and a *stable* node at $x = 0$. When $\mu = 0$, they merge into one *neutral* point at $x = 0$, then when $\mu > 0$, the two nodes have swapped positions: *unstable* at $x = 0$ and *stable* at $x = \mu$.

(iv) Run the second bifurcation demonstration **Bifurcations.demo(2)** to see how a transcritical bifurcation develops as μ changes from negative to positive values.

A ***pitchfork*** bifurcation has the canonical form $\dot{x} = \mu x - x^3$. When $\mu < 0$, there is a *stable* node at $x = 0$; when $\mu = 0$, there is *one neutral* point at $x = 0$; when $\mu > 0$, there are *three* critical points: an *unstable* node at $x = 0$ and *stable* nodes at $x = \pm\sqrt{\mu}$.

(v) Run the third bifurcation demonstration **Bifurcations.demo(3)** to see how a pitchfork bifurcation develops as μ changes from negative to positive values.

Hopf bifurcations have the canonical form $\dot{r} = r(\mu - r^2);\ \dot{\theta} = 1$. When $\mu \leq 0$, the origin is a *stable focus* with an anti-clockwise flow around it; when $\mu > 0$, there is an *unstable focus* at the origin and a stable limit cycle at $r = \sqrt{\mu}$.

(vi) There are two forms of Hopf bifurcation: *supercritical* and *subcritical*. The above definition is the ***supercritical***, or ***soft***, Hopf bifurcation. Run the fourth bifurcation demonstration **Bifurcations.demo(4)** to see how a supercritical Hopf bifurcation develops as μ changes from negative to positive values.

How can I extend my skills?

(vii) The ***subcritical***, or ***hard***, Hopf bifurcation has opposite sign: $\dot{r} = r(r^2 - \mu);\ \dot{\theta} = 1$. Implement in the **Bifurcations** module a new bifurcation to demonstrate how a hard Hopf bifurcation develops as μ changes *from positive to negative values*. As you watch, trace the trajectory of a phase point *inside* the unstable limit cycle as μ drops below zero. Discuss with a friend why some people (for example in the study of vibrations in aeroplane wings) call this a **dangerous** Hopf bifurcation?

How can I deepen my practice of the skills?

(viii) Tuna are fished in the north Atlantic Ocean at a rate h, and this leads to the following differential model for the population $x(t)$ in $\mathrm{kT/km^2}$:

$$\dot{x} = x\left(1 - \frac{x}{5}\right) - \frac{h}{0.2 + x}$$

Calculate and classify the critical points of this differential equation, then use your calculations to plot a ***bifurcation diagram*** that shows how these critical values of x depend upon an appropriate bifurcation parameter. Interpret this model, the bifurcation and your findings in physical terms and implement a dynamical model to verify your interpretation.