The Journal of Open Source Software

TikhonovFenichelReductions.jl: A systematic approach to geometric singular perturbation theory

Johannes Apelt[1]

1 Institute of Mathematics and Computer Science, University of Greifswald, Germany

DOI: 10.xxxxxx/draft

Software

- Review
- Repository
- Archive

Editor: Open Journals

Reviewers:

- @openjournals

Submitted: 01 January 1970
Published: unpublished

Summary

Singular perturbation theory is a mathematical toolbox that allows dimensionality reduction of ODE systems whose components evolve on different time scales emanating from the presence of a small parameter ε. More precisely, we obtain the *reduced system* as the limit

$$\begin{aligned} \dot{u} &= g(u,v) \\ \varepsilon \dot{v} &= h(u,v) \end{aligned} \quad \xrightarrow[\varepsilon \to 0]{} \quad \begin{aligned} \dot{u} &= g(u,v) \\ 0 &= h(u,v) \end{aligned} \tag{1}$$

as stated in Tikhonov's theorem (1952; Theorem 1.1 in Verhulst, 2007). For autonomous systems, Fenichel (1979) established a geometric singular perturbation theory (GSPT) in coordinate-free settings (see Wechselberger, 2020). Convergence properties then follow from the slow manifold $M_0 = \{(u,v) \mid h(u,v) = 0\}$, on which the reduced system is defined.

The algebraic approach to GSPT recently developed by Goeke and Walcher (and colleagues) allows to systematically find all critical parameters admitting a reduction for polynomial or rational ODE systems of the form

$$\dot{x} = f(x,\pi) = f^{(0)}(x) + \varepsilon f^{(1)}(x) + \mathcal{O}(\varepsilon^2), \quad x \in \mathbb{R}^n, \pi \in \mathbb{R}^m, \tag{2}$$

i.e. we obtain all reductions for a system with a slow-fast separation of processes instead of components as in the standard form (1), which renders this a coordinate-free approach. This can be achieved by evaluating necessary conditions for the existence of a reduction for a system as in (2) (Goeke, 2013; Goeke et al., 2015; Goeke & Walcher, 2013, 2014).

`TikhonovFenichelReductions.jl` is a `Julia` (Bezanson et al., 2017) package implementing this approach for polynomial ODE systems. Apelt & Liebscher (2025) provide a showcasing example and more detailed explanations.

Statement of need

The ad-hoc approach to singular perturbation theory requires prior knowledge about a suitable time scale separation and substantial mathematical effort to compute the reduction. The algebraic approach yields algorithmically accessible conditions for the existence of a reduction, which allows to find *all* reductions of a given polynomial ODE system using methods from computational algebra, and simplifies the computation of reduced systems (Goeke et al., 2015). `TikhonovFenichelReductions.jl` makes the required computations easily accessible (even for non-expert users) by utilizing `Oscar.jl` (Decker et al., 2025; The OSCAR Team, 2025).

The author is not aware of any publicly available implementation of the theory by Goeke and Walcher, but there exist an implementation for a related approach for computing invariant manifolds by Roberts (n.d., 1997).

Software design

`TikhonovFenichelReductions.jl` is implemented in `Julia` due to its flexibility, the use of multiple dispatch and the availability of the feature-rich computer algebra system `Oscar.jl` (Decker et al., 2025; The OSCAR Team, 2025).

Core features are the search for critical parameters admitting a reduction, so-called *Tikhonov-Fenichel Parameter Values (TFPVs)*, and the computation of the corresponding reduced systems. Crucially, this requires various computations with multiple symbolic representations (i.e. different polynomial rings, rational function fields and matrix spaces) and therefore parsing of data between the corresponding types in `Oscar.jl`, which `TikhonovFenichelReductions.jl` performs hidden away from the user. Thus, the user mostly works in an object-oriented manner with the types and methods provided.

The package is essentially designed around two types: `ReductionProblem`, which constructs all symbolic data types needed for the search of TFPVs, and `Reduction`, which holds all relevant information for the reduced system and the steps required to compute it. The latter also contains the reduced system and other information parsed to the appropriate types from `Oscar.jl`, that can be further used, e.g. for a symbolic analysis.

Features

Detailed explanations and examples are provided by Apelt & Liebscher (2025) and in the documentation.

Finding TFPVs

The package provides a method to obtain all possible TFPVs implicitly by computing a Gröbner Basis and one to find *slow-fast separations of rates*, i.e. TFPVs with some parameters set to zero. Although the former method is an extensive search, the latter is usually better suited in practice as it yields the TFPVs one is typically interested in explicitly, is more efficient, and directly outputs the corresponding slow manifolds (implicitly as affine varieties in phase space).

Computing reductions

Computing a reduction for a slow-fast separation of rates as in Theorem 1 in Goeke & Walcher (2014) requires essentially two steps. First, we need to provide a parametric representation of the slow manifold, which is given as an irreducible component of the affine variety $\mathcal{V}(f^{(0)})$. Then, we need to find a product decomposition $f^{(0)} = P\psi$, where P is a $n \times r$ matrix of rational functions and ψ is a vector of polynomials locally satisfying $\mathcal{V}(\psi) = \mathcal{V}(f^{(0)})$ and rank $P =$ rank $D\psi = r$.

With this, the reduced system in the sense of Tikhonov is given by

$$\dot{x} = \left[1_n - P(x)(D\psi(x)P(x))^{-1}D\psi(x)\right] f^{(1)}(x).$$

For convenience, the packages provides multiple heuristics, which automate the steps required to compute a reduction and allows bulk computation of multiple reductions at once.

Integration with the `Julia` ecosystem

The input system can be given as a reaction network defined with `Catalyst.jl` (Loman et al., 2023). Because the reduced systems are represented using types from `Oscar.jl`, the latter's functions can be used to aid the symbolic analysis. `Julia`'s support for metaprogramming allows to perform further tasks such as a numerical analysis without having to copy or parse code (see e.g. `TFRSimulations.jl`). For convenience, there are several methods for displaying the output, including printing as LaTeX source code via `Latexify.jl`.

Research impact statement

Time scale separations are widely used in various areas of mathematical modelling (Wechselberger, 2020). However, as far as the author is aware, the systematic approach due to Goeke and Walcher seems to be scarcely adopted, even though it comes with many advantages compared to the ad-hoc approach. This package aims to make the theory more accessible and convenient to use.

In the field of mathematical ecology (which the author is most familiar with), the theory was successfully used by Kruff et al. (2019) and more recently by Apelt & Liebscher (2025). For the model introduced in the former, the application was relatively straightforward, but more complex models as in the latter require some more work. In particular, the method for finding TFPVs that relies on the computation of a Gröbner Basis may fail for complex systems. In this case, `TikhonovFenichelReductions.jl` enables and simplifies the search for the most common TFPVs. Given the ubiquity of time scale separation techniques in this field alone (see e.g. Abbott et al., 2020; Poggiale & Auger, 2004; Revilla, 2015), the package can be a potentially useful tool for modellers.

AI usage disclosure

No generative AI tools were used in the development of this software, the writing of this manuscript, or the preparation of supporting materials.

Acknowledgements

This work was supported by a scholarship awarded by the University of Greifswald according to the “Landesgraduiertenförderungsgesetz (LGFG) MV”. I like to thank Volkmar Liebscher for his supervision and support, Sebastian Walcher and Alexandra Goeke for developing the Tikhonov-Fenichel reduction theory, Leonard Schmitz for discussing aspects of the computational algebra, and the OSCAR team for their great work and helpful replies to my questions.

References

Abbott, K. C., Ji, F., Stieha, C. R., & Moore, C. M. (2020). Fast and slow advances toward a deeper integration of theory and empiricism. *Theoretical Ecology*, *13*(1), 7–15. https://doi.org/10.1007/s12080-019-00441-x

Apelt, J., & Liebscher, V. (2025). Tikhonov-Fenichel reductions and their application to a novel modelling approach for mutualism. *Theoretical Population Biology*, 16–35. https://doi.org/10.1016/j.tpb.2025.08.004

Bezanson, J., Edelman, A., Karpinski, S., & Shah, V. B. (2017). Julia: A Fresh Approach to Numerical Computing. *SIAM Review*, *59*(1), 65–98. https://doi.org/10.1137/141000671

Decker, W., Eder, C., Fieker, C., Horn, M., & Joswig, M. (Eds.). (2025). *The Computer Algebra System OSCAR: Algorithms and Examples*. Springer Nature Switzerland. https://doi.org/10.1007/978-3-031-62127-7

Fenichel, N. (1979). Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations*, *31*(1), 53–98. https://doi.org/10.1016/0022-0396(79)90152-9

Goeke, A. (2013). *Reduktion und asymptotische Reduktion von Reaktionsgleichungen* [RWTH Aachen]. https://publications.rwth-aachen.de/record/229008

Goeke, A., & Walcher, S. (2013). Quasi-Steady State: Searching for and Utilizing Small Parameters. In A. Johann, H.-P. Kruse, F. Rupp, & S. Schmitz (Eds.), *Recent Trends in Dynamical Systems* (Vol. 35, pp. 153–178). Springer Basel. https://doi.org/10.1007/978-3-0348-0451-6_8

Goeke, A., & Walcher, S. (2014). A constructive approach to quasi-steady state reductions. *Journal of Mathematical Chemistry*, *52*(10), 2596–2626. https://doi.org/10.1007/s10910-014-0402-5

Goeke, A., Walcher, S., & Zerz, E. (2015). Determining “small parameters” for quasi-steady state. *Journal of Differential Equations*, *259*(3), 1149–1180. https://doi.org/10.1016/j.jde.2015.02.038

Kruff, N., Lax, C., Liebscher, V., & Walcher, S. (2019). The Rosenzweig–MacArthur system via reduction of an individual based model. *Journal of Mathematical Biology*, *78*(1–2), 413–439. https://doi.org/10.1007/s00285-018-1278-y

Loman, T. E., Ma, Y., Ilin, V., Gowda, S., Korsbo, N., Yewale, N., Rackauckas, C., & Isaacson, S. A. (2023). Catalyst: Fast and flexible modeling of reaction networks. *PLOS Computational Biology*, *19*(10), e1011530. https://doi.org/10.1371/journal.pcbi.1011530

Poggiale, J.-C., & Auger, P. (2004). Impact of spatial heterogeneity on a predator–prey system dynamics. *Comptes Rendus. Biologies*, *327*(11), 1058–1063. https://doi.org/10.1016/j.crvi.2004.06.006

Revilla, T. A. (2015). Numerical responses in resource-based mutualisms: A time scale approach. *Journal of Theoretical Biology*, *378*, 39–46. https://doi.org/10.1016/j.jtbi.2015.04.012

Roberts, A. J. (n.d.). *Construct invariant manifolds*. https://tuck.adelaide.edu.au/gencm.php

Roberts, A. J. (1997). Low-dimensional modelling of dynamics via computer algebra. *Computer Physics Communications*, *100*(3), 215–230. https://doi.org/10.1016/s0010-4655(96)00162-2

The OSCAR Team. (2025). *OSCAR – Open Source Computer Algebra Research system*. https://www.oscar-system.org

Tikhonov, A. N. (1952). Systems of differential equations containing small parameters in the derivatives. *Matematicheskii sbornik*, *73*(3), 575–586.

Verhulst, F. (2007). Singular perturbation methods for slow–fast dynamics. *Nonlinear Dynamics*, *50*(4), 747–753. https://doi.org/10.1007/s11071-007-9236-z

Wechselberger, M. (2020). *Geometric Singular Perturbation Theory Beyond the Standard Form*. Springer International Publishing. https://doi.org/10.1007/978-3-030-36399-4