

## Weak form of the Navier equation (Laplace + grad-div form)

Consider the strong form of linear elasticity written as

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega. \quad (1)$$

Let the boundary be decomposed as

$$\partial\Omega = \Gamma_u \cup \Gamma_t,$$

and choose test functions

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_u.$$

Multiplying the strong form by  $\mathbf{v}$  and integrating over  $\Omega$  yields

$$\int_{\Omega} \mu (\Delta \mathbf{u}) \cdot \mathbf{v} \, d\Omega + \int_{\Omega} (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) \cdot \mathbf{v} \, d\Omega + \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, d\Omega = 0. \quad (2)$$

Applying integration by parts to the Laplacian term,

$$\int_{\Omega} (\Delta \mathbf{u}) \cdot \mathbf{v} \, d\Omega = - \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega + \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{v} \, d\Gamma. \quad (3)$$

For the grad-div term, let  $\phi = \nabla \cdot \mathbf{u}$ . Then

$$\int_{\Omega} \nabla \phi \cdot \mathbf{v} \, d\Omega = - \int_{\Omega} \phi (\nabla \cdot \mathbf{v}) \, d\Omega + \int_{\partial\Omega} \phi (\mathbf{v} \cdot \mathbf{n}) \, d\Gamma. \quad (4)$$

Substituting these into the weighted equation and rearranging gives the weak form:

$$\mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega + (\lambda + \mu) \int_{\Omega} (\nabla \cdot \mathbf{u}) (\nabla \cdot \mathbf{v}) \, d\Omega = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v} \, d\Gamma \quad (5)$$

with the natural traction

$$\mathbf{t} = \mu \frac{\partial \mathbf{u}}{\partial n} + (\lambda + \mu) (\nabla \cdot \mathbf{u}) \mathbf{n}. \quad (6)$$

## Weak form with Laplace, grad-div and curl-curl operators

Consider the equivalent strong form

$$2\mu\Delta\mathbf{u} + \lambda\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega. \quad (7)$$

Let  $\mathbf{v}$  be a test function such that

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_u.$$

Multiplying by  $\mathbf{v}$  and integrating over  $\Omega$ ,

$$\int_{\Omega} \left( 2\mu\Delta\mathbf{u} + \lambda\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) + \mathbf{b} \right) \cdot \mathbf{v} \, d\Omega = 0. \quad (8)$$

For the Laplacian term,

$$\int_{\Omega} (\Delta\mathbf{u}) \cdot \mathbf{v} \, d\Omega = - \int_{\Omega} \nabla\mathbf{u} : \nabla\mathbf{v} \, d\Omega + \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial n} \cdot \mathbf{v} \, d\Gamma. \quad (9)$$

For the grad-div term,

$$\int_{\Omega} \nabla(\nabla \cdot \mathbf{u}) \cdot \mathbf{v} \, d\Omega = - \int_{\Omega} (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) \, d\Omega + \int_{\partial\Omega} (\nabla \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{n}) \, d\Gamma. \quad (10)$$

For the curl-curl term, using Stokes' identity,

$$\int_{\Omega} \nabla \times (\nabla \times \mathbf{u}) \cdot \mathbf{v} \, d\Omega = \int_{\Omega} (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \, d\Omega + \int_{\partial\Omega} (\mathbf{n} \times (\nabla \times \mathbf{u})) \cdot \mathbf{v} \, d\Gamma. \quad (11)$$

Collecting the volume terms yields the weak form

$$2\mu \int_{\Omega} \nabla\mathbf{u} : \nabla\mathbf{v} \, d\Omega + \lambda \int_{\Omega} (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) \, d\Omega - \mu \int_{\Omega} (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \, d\Omega = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_t} \mathbf{t}_3 \cdot \mathbf{v} \, d\Gamma \quad (12)$$

with the natural traction

$$\mathbf{t}_3 = 2\mu \frac{\partial\mathbf{u}}{\partial n} + \lambda(\nabla \cdot \mathbf{u})\mathbf{n} - \mu \mathbf{n} \times (\nabla \times \mathbf{u}). \quad (13)$$