Weak form of the Navier equation (Laplace + grad–div form)

Consider the strong form of linear elasticity written as

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{b} = \mathbf{0} \qquad \text{in } \Omega. \tag{1}$$

Let the boundary be decomposed as

$$\partial\Omega = \Gamma_u \cup \Gamma_t,$$

and choose test functions

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_u.$$

Multiplying the strong form by $\mathbf{v}$ and integrating over Ω yields

$$\int_\Omega \mu(\Delta\mathbf{u})\cdot\mathbf{v}\,d\Omega + \int_\Omega (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})\cdot\mathbf{v}\,d\Omega + \int_\Omega \mathbf{b}\cdot\mathbf{v}\,d\Omega = 0. \tag{2}$$

Applying integration by parts to the Laplacian term,

$$\int_\Omega (\Delta\mathbf{u})\cdot\mathbf{v}\,d\Omega = -\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,d\Omega + \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial n}\cdot\mathbf{v}\,d\Gamma. \tag{3}$$

For the grad–div term, let $\phi = \nabla\cdot\mathbf{u}$. Then

$$\int_\Omega \nabla\phi\cdot\mathbf{v}\,d\Omega = -\int_\Omega \phi\,(\nabla\cdot\mathbf{v})\,d\Omega + \int_{\partial\Omega} \phi(\mathbf{v}\cdot\mathbf{n})\,d\Gamma. \tag{4}$$

Substituting these into the weighted equation and rearranging gives the weak form:

$$\mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,d\Omega + (\lambda+\mu)\int_\Omega (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\,d\Omega = \int_\Omega \mathbf{b}\cdot\mathbf{v}\,d\Omega + \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}\,d\Gamma \tag{5}$$

with the natural traction

$$\mathbf{t} = \mu\frac{\partial\mathbf{u}}{\partial n} + (\lambda+\mu)(\nabla\cdot\mathbf{u})\mathbf{n}. \tag{6}$$

Weak form with Laplace, grad–div and curl–curl operators

Consider the equivalent strong form

$$2\mu\Delta\mathbf{u} + \lambda\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) + \mathbf{b} = \mathbf{0} \qquad \text{in } \Omega. \tag{7}$$

Let $\mathbf{v}$ be a test function such that

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_u.$$

Multiplying by $\mathbf{v}$ and integrating over Ω,

$$\int_\Omega \Big(2\mu\Delta\mathbf{u} + \lambda\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) + \mathbf{b}\Big)\cdot\mathbf{v}\, d\Omega = 0. \tag{8}$$

For the Laplacian term,

$$\int_\Omega (\Delta\mathbf{u})\cdot\mathbf{v}\, d\Omega = -\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega + \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial n}\cdot\mathbf{v}\, d\Gamma. \tag{9}$$

For the grad–div term,

$$\int_\Omega \nabla(\nabla\cdot\mathbf{u})\cdot\mathbf{v}\, d\Omega = -\int_\Omega (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\, d\Omega + \int_{\partial\Omega} (\nabla\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{n})\, d\Gamma. \tag{10}$$

For the curl–curl term, using Stokes' identity,

$$\int_\Omega \nabla\times(\nabla\times\mathbf{u})\cdot\mathbf{v}\, d\Omega = \int_\Omega (\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})\, d\Omega + \int_{\partial\Omega} (\mathbf{n}\times(\nabla\times\mathbf{u}))\cdot\mathbf{v}\, d\Gamma. \tag{11}$$

Collecting the volume terms yields the weak form

$$2\mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega + \lambda\int_\Omega (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\, d\Omega - \mu\int_\Omega (\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})\, d\Omega = \int_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_t} \mathbf{t}_3\cdot\mathbf{v}\, d\Gamma \tag{12}$$

with the natural traction

$$\mathbf{t}_3 = 2\mu\frac{\partial\mathbf{u}}{\partial n} + \lambda(\nabla\cdot\mathbf{u})\mathbf{n} - \mu\,\mathbf{n}\times(\nabla\times\mathbf{u}). \tag{13}$$