LorentzGroup.jl

Mike Savastio

Abstract

LorentzGroup.jl is a Julia package for efficiently computing actions of the Lorentz group $\mathrm{SO}(1,3)$.

1 The Lorentz Group

We review basic facts about the Lorentz group $\mathrm{SO}(1,3)$ for reference. In this note we adopt the metric signature $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, though in the Julia package this is $(1,1,1,-1)$ in order to simplify compatibility with Julia array conventions.

The group generators are given by

$$(M^{\mu\nu})^{\rho}{}_{\sigma} = \eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma} \tag{1}$$

having commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = M^{\mu\sigma}\eta^{\nu\rho} - M^{\nu\sigma}\eta^{\mu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} \tag{2}$$

Group elements can be written

$$\Lambda(\omega) = e^{\omega_{\mu\nu}M^{\mu\nu}} \tag{3}$$

Since $M^{\mu\nu}$ is anti-symmetric there are 6 parameters $\omega_{\mu\nu}$ of the 6-dimensional $\mathrm{SO}(1,3)$.

It is sometimes convenient to rewrite these in the alternate basis

$$\begin{aligned} J_i &= \frac{1}{2}\varepsilon_{ijk}M^{jk} \\ K_i &= M^{i0} \end{aligned} \tag{4}$$

which have the commutation relations

$$[J_i, J_j] = \varepsilon_{ijk}J^k \qquad [J_i, K_j] = \varepsilon_{ijk}K^k \qquad [K_i, K_j] = -\varepsilon_{ijk}J^k \tag{5}$$

and in terms of which group elements can be written

$$\Lambda(\theta, \zeta) = e^{\theta_i J^i + \omega_i K^i} \tag{6}$$

Explicitly, the generators are

$$\begin{aligned} J_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ K_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{7}$$