

Grassmann tensor renormalization group approach to (1+1)-dimensional two-color lattice QCD at finite density

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ABSTRACT: We construct a Grassmann tensor network representing the partition function of (1+1)-dimensional two-color QCD with staggered fermions. The Grassmann path integral is rewritten as the trace of a Grassmann tensor network by introducing two-component auxiliary Grassmann fields on every edge of the lattice. We introduce an efficient initial tensor compression scheme to reduce the size of initial tensors. The Grassmann bond-weighted tensor renormalization group approach is adopted to evaluate the quark number density, fermion condensate, and diquark condensate at different gauge couplings as a function of the chemical potential. Different transition behavior is observed as the quark mass is varied. We discuss the efficiency of our initial tensor compression scheme and the future application toward the corresponding higher-dimensional models.

KEYWORDS: Algorithms and Theoretical Developments, Field Theories in Lower Dimensions, Non-Zero Temperature and Density

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1 Introduction

Since numerical simulations of lattice field theories based on the Monte Carlo approach generally suffer from the sign problem, the tensor renormalization group (TRG) [1], which is a variant of real-space renormalization group on a tensor network and does not rely on any sampling methods, has become an attractive strategy for the Lagrangian formulation. Improvements in two dimensions [2–6], efficient higher dimensional algorithms [7–11], and extensions to fermion systems [12–15] have made the TRG approach more accessible to computational lattice field theories. During the last decade, TRG has been used to simulate various models not only in two dimensions but also in the higher dimensions such as the ϕ^4 theory [16–22], the Schwinger model [14, 23–26], the Gross-Neveu model [27–29], the gauge-Higgs model [30–33], and the pure Yang-Mills theory [34–37]. Toward the TRG study of the quantum chromodynamics (QCD) at finite temperature and density, it is necessary to develop a methodology to deal with the non-Abelian gauge theory in the presence of dynamical fermions. Recently, several attempts have been made for the two-dimensional QCD in the infinite-coupling limit [38] and two-color QCD with reduced staggered fermions [39]. There are also various previous studies of the (1+1)-dimensional QCD based on the Hamiltonian formalism employing the variational tensor network method such as matrix product state (MPS) [40–47]. Although the TRG and MPS approaches are different tensor network methods, they share several aspects in their numerical calculations. In particular, both

approaches require the discretization for the non-Abelian gauge fields. Therefore, findings from Lagrangian-based calculations may also yield useful information for the Hamiltonian-based approach, and vice versa.

Two-color QCD, the $SU(2)$ Yang-Mills theory coupled with fermions in the fundamental representation of the gauge group, serves as a nice primary target for the tensor network approach. It has fewer degrees of freedom than the three-color QCD and the validity of the numerical results can be verified through comparison with the Monte Carlo method, particularly in four dimensions; two-color QCD has been investigated as an alternative to the three-color QCD because its fermion determinant is positive when the number of quark flavors is even. This enables Monte Carlo simulations at finite density in contrast to the three-color case [48, 49]. Two-color QCD shares several properties with three-color QCD such as the spontaneous breaking of chiral symmetry at low density with a restoration of the symmetry at larger density. In addition, there is an instability of the Fermi sphere against the formation of diquark condensate at sufficiently large density in both theories. Although diquarks in two-color QCD are color singlets, which differs from the case in three-color QCD, lattice simulations of two-color QCD are expected to provide hints about the nature of deconfined phase and the mechanism of condensate formation in dense three-color QCD.

In this work, we construct a Grassmann tensor network representation for the partition function of $(1+1)$ -dimensional two-color QCD with staggered fermions and apply the Grassmann bond-weighted TRG (BTRG) algorithm [6, 50] to evaluate the expectation values of several physical observables. As an advantage of the Grassmann tensor network formulation, we can directly deal with the fermionic degrees of freedom without introducing pseudo-fermion. We focus on the finite density regime in the thermodynamic and vanishing temperature limits and investigate whether the TRG computation can capture the effect of finite gauge coupling, quark mass, and diquark source term. For the four-dimensional two-color QCD, the phase diagram in the T - μ - m space, with the temperature T , chemical potential μ , and quark mass m , has been intensively studied [51–60]. With a view to future applications of the TRG approach to the four-dimensional theory, we particularly compute the number density, chiral condensate, and diquark condensate, commonly used to investigate the phase structure. In this study, the chiral and diquark condensates are evaluated by explicitly breaking the $U(1)_A$ and $U(1)_V$ symmetries because continuous symmetry is not spontaneously broken in two dimensions [61, 62]. We confirm that their behavior is consistent with each other. We also see that the number density does not saturate in regions of larger chemical potential as the gauge interaction is weakened, approaching the continuum limit, in other words. Our work is the first TRG study of the non-Abelian gauge theory with finite gauge coupling in the presence of the fermions at finite density.

This paper is organized as follows. In section 2, we introduce the tensor network representation of the lattice theory. In section 3, we briefly review BTRG and describe the initial tensor compression scheme facilitating practical calculations. In section 4, we present numerical results in the infinite coupling limit and the finite coupling regime. Section 5 is devoted to the conclusion and outlook.

2 Tensor network representation of $SU(N)$ Yang-Mills theory with staggered fermion

2.1 Lattice theory

We consider a $(1+1)$ -dimensional $SU(N)$ Yang-Mills theory coupled with staggered fermion defined on a square lattice Λ with volume $V = L^2$. The action is given by

$$S = S_f + S_g \quad (2.1)$$

with

$$S_f = \sum_{n \in \Lambda, \nu=1,2} \frac{p_\nu(n)}{2} \left[e^{\mu \delta_{\nu,2}} \bar{\chi}(n) U_\nu(n) \chi(n + \hat{\nu}) - e^{-\mu \delta_{\nu,2}} \bar{\chi}(n + \hat{\nu}) U_\nu^\dagger(n) \chi(n) \right] + m \sum_n \bar{\chi}(n) \chi(n), \quad (2.2)$$

$$S_g = -\frac{\beta}{N} \sum_n \text{Re Tr } U_1(n) U_2(n + \hat{1}) U_1^\dagger(n + \hat{2}) U_2^\dagger(n), \quad (2.3)$$

where each lattice site is labeled by $n = (n_1, n_2)$ with $n_\nu = 1, \dots, L$. The lattice spacing a is always set to $a = 1$. The N -component staggered fermions are denoted by $\chi(n)$ and $\bar{\chi}(n)$ with the mass m and chemical potential μ . We define the staggered phase function as $p_1(n) = 1$ and $p_2(n) = (-1)^{n_1}$. The link variables $U_\nu(n) \in SU(N)$ live on the link from the site n to $n + \hat{\nu}$ and the inverse gauge coupling is represented by β . The partition function is given by a Euclidean path integral

$$Z = \int \mathcal{D}U \mathcal{D}\chi \mathcal{D}\bar{\chi} e^{-S}, \quad (2.4)$$

where $\mathcal{D}U \equiv \prod_n dU_1(n) dU_2(n)$ and dU is the Haar measure of $SU(N)$. The Grassmann path integral measure is defined by $\mathcal{D}\chi \mathcal{D}\bar{\chi} \equiv \prod_n \prod_{c=1}^N d\chi_c(n) d\bar{\chi}_c(n)$. We always assume the periodic boundary conditions for link variables and the (anti-)periodic boundary conditions for staggered fermions in the $\hat{1}$ ($\hat{2}$) direction.

2.2 Tensor network representation

We begin with considering the fermionic sector. Expressing eq. (2.4) as

$$Z = \int \mathcal{D}U Z_f[U] e^{-S_g}, \quad (2.5)$$

where

$$Z_f[U] = \int \mathcal{D}\chi \mathcal{D}\bar{\chi} e^{-S_f}, \quad (2.6)$$

we firstly derive a tensor network representation of $Z_f[U]$. Following the formalism in ref. [15], we introduce two N -component auxiliary Grassmann fields $\eta_\nu(n)$ and $\zeta_\nu(n)$ on every link of the lattice. We can decompose the fermion hopping terms in eq. (2.2) with these auxiliary

Grassmann fields via

$$\begin{aligned} & \exp \left[-\frac{p_\nu(n)}{2} e^{\mu\delta_{\nu,2}} \bar{\chi}(n) U_\nu(n) \chi(n+\hat{\nu}) \right] \\ &= \int_{\bar{\eta}_\nu(n), \eta_\nu(n)} \exp \left[-\bar{\chi}(n) \eta_\nu(n) + \frac{p_\nu(n)}{2} e^{\mu\delta_{\nu,2}} \bar{\eta}_\nu(n) U_\nu(n) \chi(n+\hat{\nu}) \right], \end{aligned} \quad (2.7)$$

$$\begin{aligned} & \exp \left[\frac{p_\nu(n)}{2} e^{-\mu\delta_{\nu,2}} \bar{\chi}(n+\hat{\nu}) U_\nu^\dagger(n) \chi(n) \right] \\ &= \int_{\bar{\zeta}_\nu(n), \zeta_\nu(n)} \exp \left[\bar{\chi}(n+\hat{\nu}) U_\nu^\dagger(n) \bar{\zeta}_\nu(n) - \frac{p_\nu(n)}{2} e^{-\mu\delta_{\nu,2}} \zeta_\nu(n) \chi(n) \right]. \end{aligned} \quad (2.8)$$

In eqs. (2.7) and (2.8), we introduced the following short-hand notation,

$$\int_{\bar{\theta}, \theta} = \prod_{c=1}^N \int d\bar{\theta}_c d\theta_c e^{-\bar{\theta}_c \theta_c}, \quad (2.9)$$

with the N -component Grassmann variables θ and $\bar{\theta}$. By these decompositions, it is easy to integrate out the original Grassmann fields $\chi(n)$ and $\bar{\chi}(n)$ at each lattice site independently. The integration at the site n results in a Grassmann tensor $\mathcal{F}_n$ such as

$$\mathcal{F}_n[U] = \sum_{x,t,x',t'} (F_n)_{xtx't'} [U_1(n-\hat{1}), U_2(n-\hat{2})] X^x T^t \bar{X}^{x'} \bar{T}^{t'}. \quad (2.10)$$

Due to eqs. (2.7) and (2.8), the coefficient tensor $(F_n)_{xtx't'}$ depends on the link variables $U_1(n-\hat{1})$ and $U_2(n-\hat{2})$ and so does $\mathcal{F}_n$. Since we are dealing with the N -component fermion theory consisting of two types of hopping terms, the subscripts x, t, x', t' take their values on $\{0, 1\}^{2N}$. Therefore, the coefficient tensor $(F_n)_{xtx't'}$ is a rank-4 complex-valued tensor whose bond dimension is 2^{2N} . In the right-hand side of eq. (2.10), X^x abbreviates $\eta_{x,1}^{x_1} \cdots \eta_{x,N}^{x_N} \zeta_{x,1}^{x_{N+1}} \cdots \zeta_{x,N}^{x_{2N}}$ with $x = (x_1, \dots, x_{2N})$, as well as T^t . Note that X^x (T^t) resides on the positive $\hat{1}$ ($\hat{2}$) link connected to site n as shown in figure 1 (a). Similarly, $\bar{X}^{x'} = \bar{\zeta}_{x',N}^{x'_{2N}} \cdots \bar{\zeta}_{x',1}^{x'_{N+1}} \bar{\eta}_{x',N}^{x'_N} \cdots \bar{\eta}_{x',1}^{x'_1}$ with $x' = (x'_1, \dots, x'_{2N})$, as well as $\bar{T}^{t'}$, where $\bar{X}^{x'}$ ($\bar{T}^{t'}$) resides on the negative $\hat{1}$ ($\hat{2}$) link connected to site n . See appendix A for the derivation of F . We can restore $Z_f[U]$ in eq. (2.6) by contracting a Grassmann tensor network generated by $\mathcal{F}_n$:

$$Z_f[U] = \text{gTr} \left[\prod_n \mathcal{F}_n[U] \right]. \quad (2.11)$$

Here, “gTr” denotes the integration over all auxiliary Grassmann fields. The graphical representation of eq. (2.11) is illustrated in figure 1 (b).

We now move on to the gauge sector with $\beta \neq 0$. We need to discretize the gauge group integration to derive the tensor network representation of eq. (2.5). In this study, we use the method in ref. [34], which approximates a group integration by an average of integrand evaluated using K random $\text{SU}(N)$ matrices picked uniformly from the group manifold:

$$\int dU f(U) \simeq \frac{1}{K} \sum_{i=1}^K f(U_i). \quad (2.12)$$

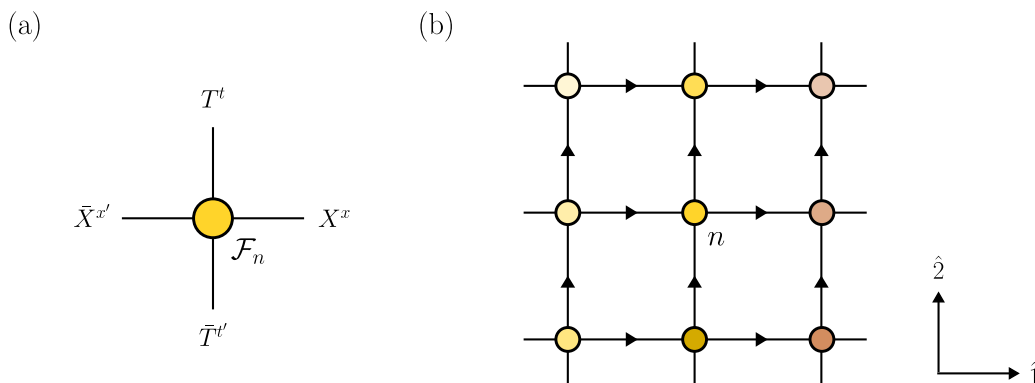


Figure 1. (a) Graphical representation of $\mathcal{F}_n$ and the auxiliary Grassmann fields that the local tensor carries. (b) Tensor network representation of $Z_f[U]$ in eq. (2.6). Notice that every local tensor is distinct from each other because $\mathcal{F}_n$ depends on the U_ν residing on the links connected to it.

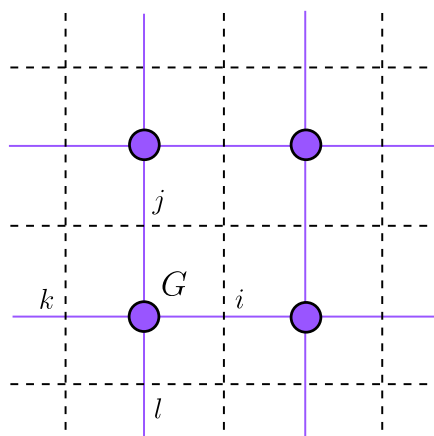


Figure 2. Tensor network representation for the partition function of pure $SU(N)$ Yang-Mills theory defined on a square lattice, which is indicated by the dashed lines.

We assign a tensor to each plaquette of the square lattice

$$G_{ijkl} \equiv \frac{1}{K^2} e^{(\beta/N) \text{Re Tr}(U_i U_j^\dagger U_k^\dagger U_l)}, \quad (2.13)$$

where the indices i, j, k, l range from 1 to K . Each of them corresponds to one of the four links of the plaquette and its value specifies which random matrix in the set $\hat{U} = \{U_1, U_2, \dots, U_K\}$ ¹ is being substituted into the right-hand side of eq. (2.13). The partition function of pure $SU(N)$ Yang-Mills theory defined on a square lattice is given by the trace of a tensor network composed of G , as shown in figure 2.

Now, we combine $\mathcal{F}_n$ and G on the plaquette with n being its top right corner to form the initial tensor (see figure 3 (a)):

$$\mathcal{T}_n = \mathcal{F}_n \cdot G. \quad (2.14)$$

¹In this study, the group integration of every link variable is approximated using the same set of random matrices $\hat{U}$. One can use more than one set of matrices for different links of the lattice at the cost of increasing the number of distinct initial tensors.

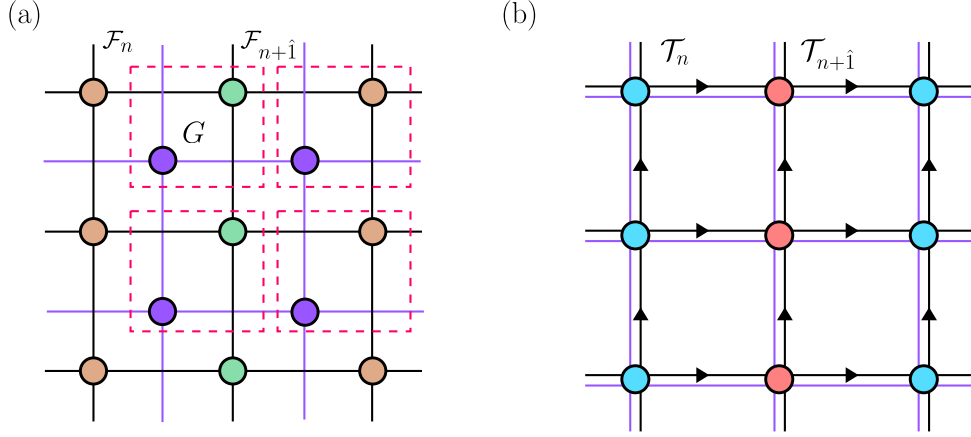


Figure 3. (a) $\mathcal{F}_n$ and G are combined to form a new Grassmann tensor. (b) Tensor network representation for the partition function of the full theory, which is composed of two initial tensors with bond dimension $2^{2N}K$.

We then regard $\mathcal{T}_n$ as a new Grassmann tensor whose coefficient tensor is given by

$$(\mathcal{T}_n)_{xtx't'} = (F_n)_{x_f t_f x'_f t'_f} [t_g, x_g] \cdot G_{x_g t_g x'_g t'_g}, \quad (2.15)$$

where the subscripts in F_n are marked with f and those in G are marked with g . In the left-hand side of eq. (2.15), we defined a super index q by $q = (q_f, q_g)$ with $q = x, t, x', t'$. Therefore, the bond dimension of $\mathcal{T}_n$ is $2^{2N}K$.² The partition function in eq. (2.5) is approximately expressed by using the fundamental tensor $\mathcal{T}_n$ as

$$Z \simeq Z(K) = \text{gTr} \left[\prod_n \mathcal{T}_n \right], \quad (2.16)$$

as illustrated in figure 3 (b). Note that “gTr” in eq. (2.16) means the summation over all subscripts marked by g and the integration over all auxiliary Grassmann fields.

We finally remark that the infinite coupling limit $\beta \rightarrow 0$ can be easily taken within the current Grassmann tensor network formulation. In this limit, we can perform the $\text{SU}(N)$ group integration exactly for all link variables on the lattice because the dependence of any $U_\nu(n)$ now appears in only one local tensor in our tensor network representation:

$$Z_{\beta \rightarrow 0} = \int \mathcal{D}U Z_f[U] = \text{gTr} \left[\prod_n \mathcal{F}'_n \right], \quad (2.17)$$

where

$$\mathcal{F}'_n = \prod_\nu \int dU_\nu \mathcal{F}_n[U] = \sum_{x,t,x',t'} \left(\int dU_1 dU_2 (F_n)_{xtx't'} [U_1, U_2] \right) X^x T^t \bar{X}^{x'} \bar{T}^{t'}. \quad (2.18)$$

²We comment on a Grassmann tensor network formulation of the $(1+1)$ -dimensional $\text{SU}(2)$ Yang-Mills theory with the standard staggered fermion recently discussed in ref. [39]. Although their formulation results in the tensor network whose bond dimension is $2^8 K$, our construction results in $2^4 K$. This is because our derivation introduces the N -component auxiliary fermions for forward and backward hopping terms as in eqs. (2.7) and (2.8). On the other hand, ref. [39] introduces the N^2 -component auxiliary fermions for each hopping term.

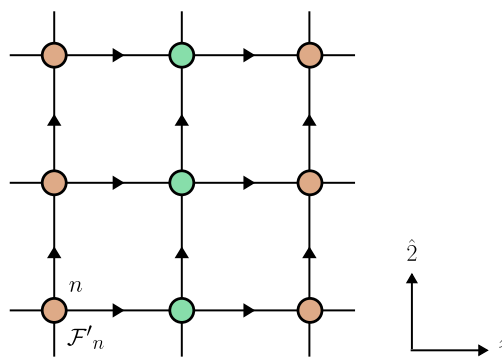


Figure 4. Tensor network representation of $Z_{\beta \rightarrow 0}$ in eq. (2.17).

The bond dimension of $\mathcal{F}'_n$ is 2^{2N} . The graphical representation of eq. (2.17) is shown in figure 4.

3 Algorithm

3.1 A quick review of BTRG

The coarse-graining transformation of a tensor network can be facilitated using tensor renormalization group (TRG) algorithms. In TRG, the low-rank approximation of tensors, which is based on the singular value decomposition (SVD), is used to perform tensor contractions. In this study, the bond-weighted tensor renormalization group (BTRG) algorithm [6] is employed. Bond weights, which are some power k of the singular values from the SVD in the previous RG iteration, are introduced on the edges of the tensor network. It was suggested and confirmed in the case of the two-dimensional Ising model [6] and massless free Wilson fermion [50] that the optimal choice for the hyperparameter k is $-1/2$ for square tensor networks. The hyperparameter k is always set to be $-1/2$ in this study, and D is the bond dimension cutoff of the BTRG algorithm, which usually depends on the initial bond dimension.

3.2 Initial tensor compression

As we mentioned, the bond dimension of the initial tensors in the Grassmann tensor network representing the partition function of the full theory is $2^{2N}K$. Although we only investigate the two-color ($N = 2$) case numerically in this study, the initial bond dimension is $16K$. This implies that a very large D is inevitable for accurate TRG results.

To tackle this problem, we propose an efficient tensor compression scheme that aims to find an accurate low-rank approximation for the initial tensors. The central idea is to insert a pair of squeezers, which approximates the identity, on every bond of the tensor network. By contracting an initial tensor with the squeezers connected to it, its bond dimension can be reduced. Now, we explain how to construct the two squeezers on a bond connecting $\mathcal{T}_{n+\hat{1}}$ and $\mathcal{T}_n$. The procedure is graphically summarized in figure 5.

We first define a matrix notation for the coefficient tensors T_n and $T_{n+\hat{1}}$ as

$$\begin{aligned} (M_A)_{x'(xtt')} &= (T_{n+\hat{1}})_{xtx't'} (-1)^{f_{x'}(f_x+f_t)}, \\ (M_B)_{(tx't')x} &= (T_n)_{xtx't'} (-1)^{f_x(f_t+f_{x'}+f_{t'})}. \end{aligned} \quad (3.1)$$

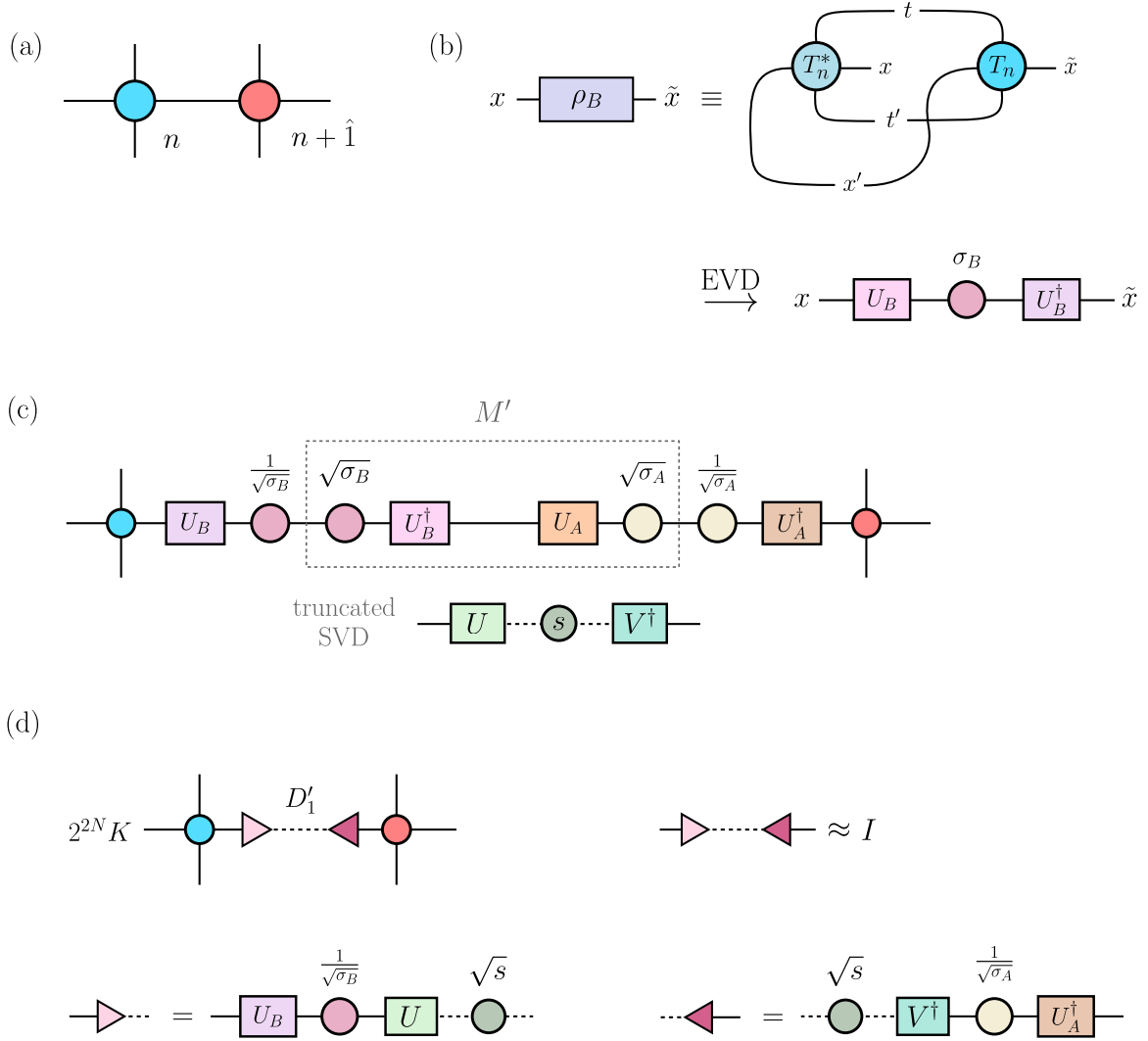


Figure 5. Procedure of constructing a pair of squeezers. (a) Two adjacent Grassmann tensors $\mathcal{T}_n$ and $\mathcal{T}_{n+1}$. (b) From the Grassmann tensor $\mathcal{T}_n$, we define a Hermitian matrix ρ_B , whose EVD gives us a unitary matrix U_B and the corresponding eigenvalue σ_B . We repeat the same procedure for $\mathcal{T}_{n+1}$ and obtain a unitary matrix U_A and the corresponding eigenvalue σ_A . (c) Inserting two pairs of invertible matrices, the truncated SVD is performed. (d) The truncated SVD in (c) defines the pair of squeezers.

In eq. (3.1), we introduced a Grassmann parity function f_q for a super index $q = (q_f, q_g)$ as

$$f_q = \sum_{i=1}^{2N} q_{f,i} \mod 2, \quad (3.2)$$

which diagnoses the Grassmann parity of Q^{q_f} ($Q^{q_f} = X^{x_f}, T^{t_f}, \bar{X}^{x'_f}, \bar{T}^{t'_f}$). Therefore, Grassmann fields will not appear explicitly in the following discussion, and the corresponding Grassmann algebra has already been encoded by the sign factors in eq. (3.1). Under this notation, the question becomes a low-rank approximation of the matrix $M_B M_A$. An obvious solution is a truncated SVD. In this study, a more computationally economical approach is considered.

We insert an identity $I = L_B^{-1} L_B R_A R_A^{-1}$ in the middle of $M_B M_A$ and perform an SVD on $M' \equiv L_B R_A = U s V^\dagger$. The resulting expression $M_B L_B^{-1} U s V^\dagger R_A^{-1} M_A$ is a compact SVD of $M_B M_A$ when

$$E^\dagger E = I \quad \text{with} \quad E = M_B L_B^{-1} \quad (3.3)$$

and

$$W^\dagger W = I \quad \text{with} \quad W^\dagger = R_A^{-1} M_A. \quad (3.4)$$

To find an L_B satisfying eq. (3.3), we define a Hermitian matrix $\rho_B = M_B^\dagger M_B$ which corresponds to the coefficient tensor of $\mathcal{T}_n^\dagger \mathcal{T}_n$. The eigenvalue decomposition (EVD) on ρ_B gives a unitary matrix U_B , which diagonalizes ρ_B and the corresponding eigenvalues σ_B . Then, $L_B^{-1} = U_B \frac{1}{\sqrt{\sigma_B}}$ and $L_B = \sqrt{\sigma_B} U_B^\dagger$. Similarly, another Hermitian matrix $\rho_A = M_A M_A^\dagger$ which corresponds to the coefficient tensor of $\mathcal{T}_{n+\hat{1}} \mathcal{T}_{n+\hat{1}}^\dagger$ is constructed and we perform an EVD on ρ_A . Then, $R_A^{-1} = \frac{1}{\sqrt{\sigma_A}} U_A^\dagger$ satisfies eq. (3.4) with $R_A = U_A \sqrt{\sigma_A}$. To obtain a low-rank approximation of $M_B M_A$, we only keep the largest D' singular values and vectors in the decomposition $M' = U s V^\dagger$. $D' \leq 2^{2N} K$ is the smallest integer satisfying the following condition

$$\frac{\sum_{y=1}^{D'} s_y^2}{\sum_{y=1}^{2^{2N} K} s_y^2} \geq r, \quad (3.5)$$

where $r \leq 1$ is a parameter of this compression scheme. The number of singular values retained from the even sector of M is denoted as $D'_{\text{even}} \leq 2^{2N-1} K$. We then define the two squeezers as $P_B = U_B \frac{1}{\sqrt{\sigma_B}} U \sqrt{s}$ and $Q_A = \sqrt{s} V^\dagger \frac{1}{\sqrt{\sigma_A}} U_A^\dagger$. Without truncation, $P_B Q_A = I$. The size of P_B and Q_A after truncation are $2^{2N} K \times D'$ and $D' \times 2^{2N} K$ respectively. The matrix representation of the compressed tensors at site n and $n + \hat{1}$ are $M_B P_B$ and $Q_A M_A$ respectively. One can then read off the corresponding coefficient tensors T'_n and $T'_{n+\hat{1}}$.

Through this procedure, the bond dimension being considered is reduced from $2^{2N} K$ to D' , which depends on the parameter r according to eq. (3.5). As shown in figure 6, there are four different types of bonds in the Grassmann tensor network representing the partition function. One can repeat the above steps to construct a pair of squeezers for each of the remaining types of bonds and contract the initial tensor at each lattice with the four squeezers connected to it to obtain a compressed initial tensor.³

4 Numerical results

By expressing the partition function (2.4) in terms of the trace of a Grassmann tensor network, one can compute the free energy density $f = \ln Z / V$ directly with TRG algorithms. Then, the expectation value of a physical observable $\langle O \rangle \equiv Z^{-1} \int \mathcal{D}U \mathcal{D}\chi \mathcal{D}\bar{\chi} O e^{-S}$ is given by the partial derivative of f .⁴ Two physical observables of interest in this study are the

³The bond dimension of the four indices of the compressed initial tensor should be, in general, different from each other. It is because the bond dimension after compression D' is determined by eq. (3.5), and the singular value spectrum s is different for the four different types of bonds in the tensor network.

⁴We note that the expectation value can also be expressed as the trace of a tensor network composed of an impurity tensor on some lattice sites and the earlier defined tensor $\mathcal{T}_n$ on the remaining sites [63, 64].

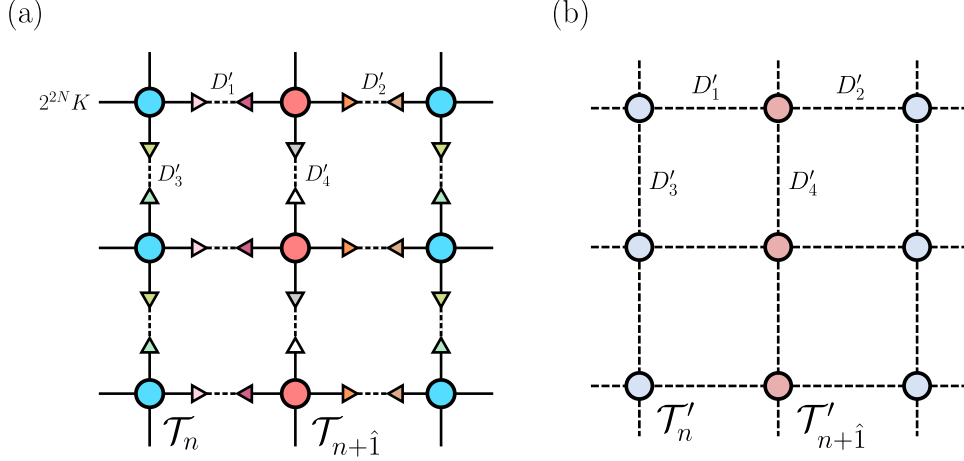


Figure 6. (a) Insertion of pairs of squeezers on every bond. (b) Compressed Grassmann tensor network.

quark number density defined as

$$\langle n \rangle = \frac{\partial f}{\partial \mu}, \quad (4.1)$$

and the fermion condensate, which is defined as

$$\langle \bar{\chi} \chi \rangle = \frac{\partial f}{\partial m}. \quad (4.2)$$

In this study, the partial derivatives in eqs. (4.1) and (4.2) are evaluated by the forward difference:

$$\langle n \rangle \simeq \frac{f(\mu + \Delta\mu) - f(\mu)}{\Delta\mu}, \quad (4.3)$$

$$\langle \bar{\chi} \chi \rangle \simeq \frac{f(m + \Delta m) - f(m)}{\Delta m}. \quad (4.4)$$

In addition to the number density and fermion condensate, we also investigate the formation of diquark condensate. When we compute the diquark condensate, we add a diquark source term to the action (2.1) as

$$S' = S + \frac{\lambda}{2} \sum_n \left[\chi^T(n) \sigma_2 \chi(n) + \bar{\chi}(n) \sigma_2 \bar{\chi}^T(n) \right], \quad (4.5)$$

where λ is a real parameter controlling the magnitude of the diquark source term, and the superscript T means a transpose. The partition function and free energy density evaluated with S' now depend on λ . It is straightforward to include this diquark source term in our Grassmann tensor network representation because it only contains single-site terms. See appendix A for the derivation of the initial tensor elements with the new action S' . The expectation value of the diquark condensate defined as $\chi\chi \equiv \sum_n (\chi^T \sigma_2 \chi + \bar{\chi} \sigma_2 \bar{\chi}^T) / 2V$ can be computed by

$$\langle \chi\chi \rangle \equiv \frac{1}{2V} \int \mathcal{D}U \mathcal{D}\chi \mathcal{D}\bar{\chi} \sum_n (\chi^T \sigma_2 \chi + \bar{\chi} \sigma_2 \bar{\chi}^T) e^{-S'} = \frac{\partial f}{\partial \lambda}. \quad (4.6)$$

The two-color lattice QCD theory under consideration has the symmetry $U(1)_V \times U(1)_A$ at a finite chemical potential μ , in the vanishing λ limit and chiral limit $m = 0$. The $U(1)_V$ and $U(1)_A$ symmetry correspond to the baryon number and axial charge conservation, respectively. A finite quark mass $m \neq 0$ breaks the $U(1)_A$ symmetry: $\chi \rightarrow e^{i\alpha\epsilon(n)}\chi$, $\bar{\chi} \rightarrow \bar{\chi}e^{i\alpha\epsilon(n)}$ ($\alpha \in \mathbb{R}$) explicitly, and a finite λ breaks the $U(1)_V$ symmetry: $\chi \rightarrow e^{i\alpha'}\chi$, $\bar{\chi} \rightarrow \bar{\chi}e^{-i\alpha'}$ ($\alpha' \in \mathbb{R}$), explicitly. Note that $\epsilon(n)$ is defined as $\epsilon(n) = (-1)^{n_1+n_2}$, which plays the similar role of γ_5 in the staggered fermion theory. Therefore, these observables are intensively studied in the four-dimensional theory [51–60].

However, since no spontaneous breaking of continuous symmetry happens in two dimensions [61, 62], we expect $\lim_{m \rightarrow 0} \lim_{V \rightarrow \infty} \langle \bar{\chi}\chi \rangle = 0$, and $\lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \langle \chi\chi \rangle = 0$. To illustrate that the behavior of the aforementioned physical observables which reveals the phase structure of the theory in higher dimensions can be properly reproduced by the TRG approach, we always set $m > 0$ and/or $\lambda > 0$ in this study,⁵ which explicitly breaks the symmetry of this model and allows a finite value of $\langle \bar{\chi}\chi \rangle$ and $\langle \chi\chi \rangle$. In particular, the diquark condensate is evaluated by

$$\langle \chi\chi \rangle \simeq \frac{f(\lambda + \Delta\lambda) - f(\lambda)}{\Delta\lambda}. \quad (4.7)$$

4.1 Infinite coupling limit

As mentioned in section 2.2, the gauge group integration can be integrated exactly in the infinite coupling limit, and the bond dimension of the initial tensors is $2^4 = 16$. Therefore, the initial tensor compression scheme introduced in section 3.2 is not employed at $\beta = 0$. In the following, we set a bond dimension cutoff $D = 84$, which suffices to suppress the finite- D effect.

Figure 7 shows the quark number density, fermion condensate, and diquark condensate with $m = 0.1$ and on a lattice volume $V = 2^{20}$. All these quantities do not depend on the chemical potential up to some point, μ_{c1} . These are the characteristic features of the so-called Silver-Blaze phenomena [65]. Since the Silver-Blaze phenomena take place only in the thermodynamic and zero-temperature limits, the lattice volume $V = 2^{20}$ is sufficiently large to obtain these limits. We observe the Silver-Blaze region where $\langle n \rangle = 0$ from $\mu = 0$ to $\mu_{c1} \approx 0.22$. As μ further increases, we see an intermediate phase, which extends over a finite region of chemical potential $\mu_{c1} < \mu < \mu_{c2}$, with $\mu_{c2} \approx 0.46$, characterized by $0 < \langle n \rangle < 2$ and non-zero $\langle \chi\chi \rangle$. As $\mu > \mu_{c2}$, $\langle n \rangle/2$ saturates to one, the maximum according to the Pauli exclusion principle on the lattice, and $\langle \chi\chi \rangle$ is suppressed to some values very close to zero. On the other hand, the fermion condensate $\langle \bar{\chi}\chi \rangle$ takes a constant finite value in the Silver-Blaze region and decreases in the intermediate phase. When $\mu > \mu_{c2}$, $\langle \bar{\chi}\chi \rangle$ is reduced to some very small values as $\langle \chi\chi \rangle$. The qualitative behavior of the observables computed by BTRG at finite m and/or λ is similar to that reported in the previous mean-field theory [54].

For a larger quark mass $m = 1$ and the same lattice volume $V = 2^{20}$, a sharp transition happens, and the intermediate phase becomes a very narrow region in μ as shown in figure 8. We see $\mu_{c1} \approx 0.982$ and $\mu_{c2} \approx 1$. Apart from this difference, the qualitative behavior of $\langle n \rangle$, $\langle \bar{\chi}\chi \rangle$ and $\langle \chi\chi \rangle$ at $m = 1$, $\beta = 0$ is similar to that at $m = 0.1$. Particularly, as shown in the inset of figure 8, there is still no discontinuity of physical quantities observed for $m = 1$ in

⁵The quark number density $\langle n \rangle$ and the fermion condensate $\langle \bar{\chi}\chi \rangle$ are calculated at $\lambda = 0$ in this study.

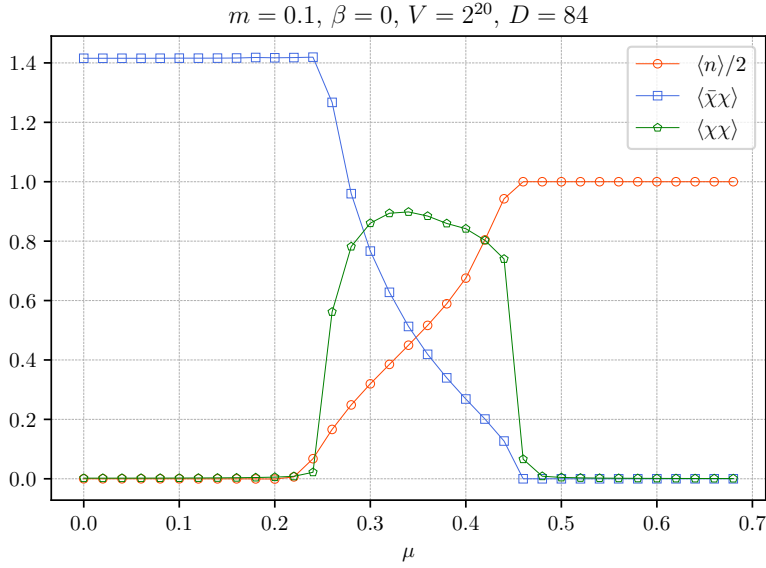


Figure 7. Quark number density $\langle n \rangle$, fermion condensate $\langle \bar{\chi}\chi \rangle$ and diquark condensate $\langle \chi\chi \rangle$ as a function of chemical potential μ at $m = 0.1$, $\beta = 0$, in the thermodynamic limit. The bond dimension in the calculations is $D = 84$. To evaluate the numerical differences in eqs. (4.3), (4.4), and (4.7), we set $\Delta\mu = 0.04$, $\Delta m = 10^{-4}$, and $\lambda = \Delta\lambda = 10^{-4}$.

the thermodynamic limit. It is fair to identify the transition in the intermediate phase as a crossover, instead of a first-order phase transition.

We also show the number density $\langle n \rangle$ and diquark condensate $\langle \chi\chi \rangle$ as a function of μ , for $m = 0.1$ and $m = 1$, at different lattice volumes in figure 9. For both quark masses, the thermodynamic limit is reached when $V = 2^{20}$. In the intermediate phase, $\langle \chi\chi \rangle$ increases with the lattice volume until the thermodynamic limit is achieved. As shown in the inset of figure 9(d), the negative value of $\langle \chi\chi \rangle$ observed in the intermediate phase might indicate that the current choice of bond dimension ($D = 84$) is not large enough for the calculations in small lattice volume.

4.2 Finite- β regime

For $\beta > 0$, the initial tensor compression scheme is applied before the BTRG calculation. Therefore, we first demonstrate the efficiency of our compression scheme in this section. In table 1, we can see how the four compressed bond dimensions D' vary with the ratio parameter r defined in eq. (3.5) at $m = 0.1$, $\beta = 1.6$, $\mu = 0.4$, $\lambda = 0$, and $K = 14$ as a representative. When $r = 1$, no compression is made, and we have an original bond dimension that is equal to $16K = 224$ with $K = 14$. For an r close enough to one, e.g., $r = 0.9999$, the bond dimension of the initial tensors is reduced from 224 to less than half of its original value after compression. The number of tensor elements is only 5.3% of the original one in this case. We also present another example in table 2, where β is changed to 0.8. It confirms that our initial tensor compression scheme is more efficient at a smaller β . In the following, we always set $r = 0.9999$.

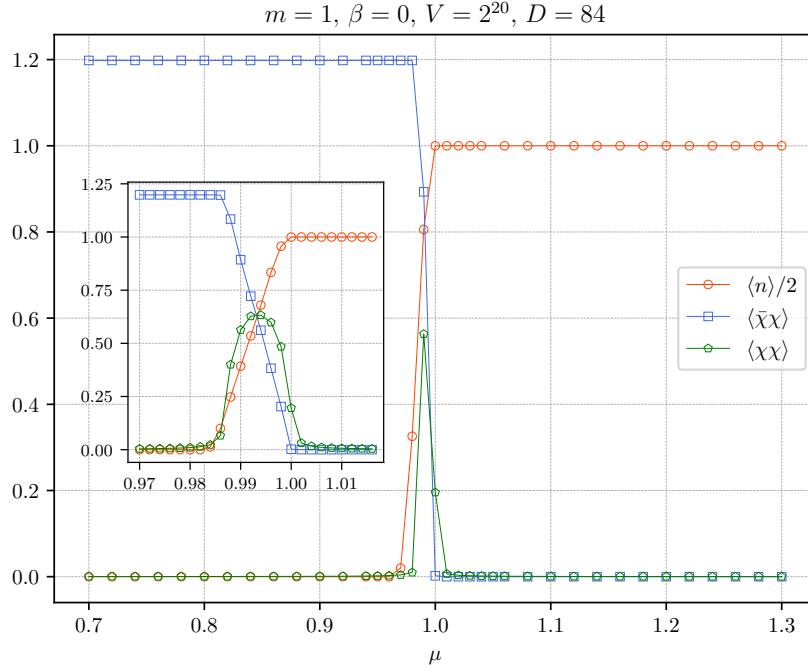


Figure 8. Quark number density $\langle n \rangle$, fermion condensate $\langle \bar{\chi}\chi \rangle$ and diquark condensate $\langle \chi\chi \rangle$ as a function of chemical potential μ at $m = 1$, $\beta = 0$, in the thermodynamic limit. The bond dimension in the calculations is $D = 84$. To evaluate the numerical differences in eqs. (4.3), (4.4), and (4.7), we set $\Delta\mu = 0.02$, $\Delta m = 10^{-4}$, and $\lambda = \Delta\lambda = 10^{-4}$. The inset shows the three quantities in the intermediate phase, where $\langle n \rangle$ is evaluated using a finer $\Delta\mu = 0.004$.

r	D'_1	D'_2	D'_3	D'_4	compression rate
1	224	224	224	224	100%
0.99999	148	148	143	143	17.8%
0.99995	122	122	118	118	8.23%
0.9999	110	110	105	105	5.30%
0.9995	80	80	79	79	1.59%
0.999	70	70	67	67	0.874%
0.99	35	35	33	33	0.0530%

Table 1. Efficiency of our initial tensor compression scheme. We set $m = 0.1$, $\beta = 1.6$, $\mu = 0.4$, $\lambda = 0$ and $K = 14$ as a representative. As shown in figure 6, D'_1 and D'_2 denote the spatial bond dimensions and D'_3 and D'_4 denote the temporal ones. The compression rate in the last column is measured as the number of elements in $\mathcal{T}'_n$ divided by the number of elements in $\mathcal{T}_n$.

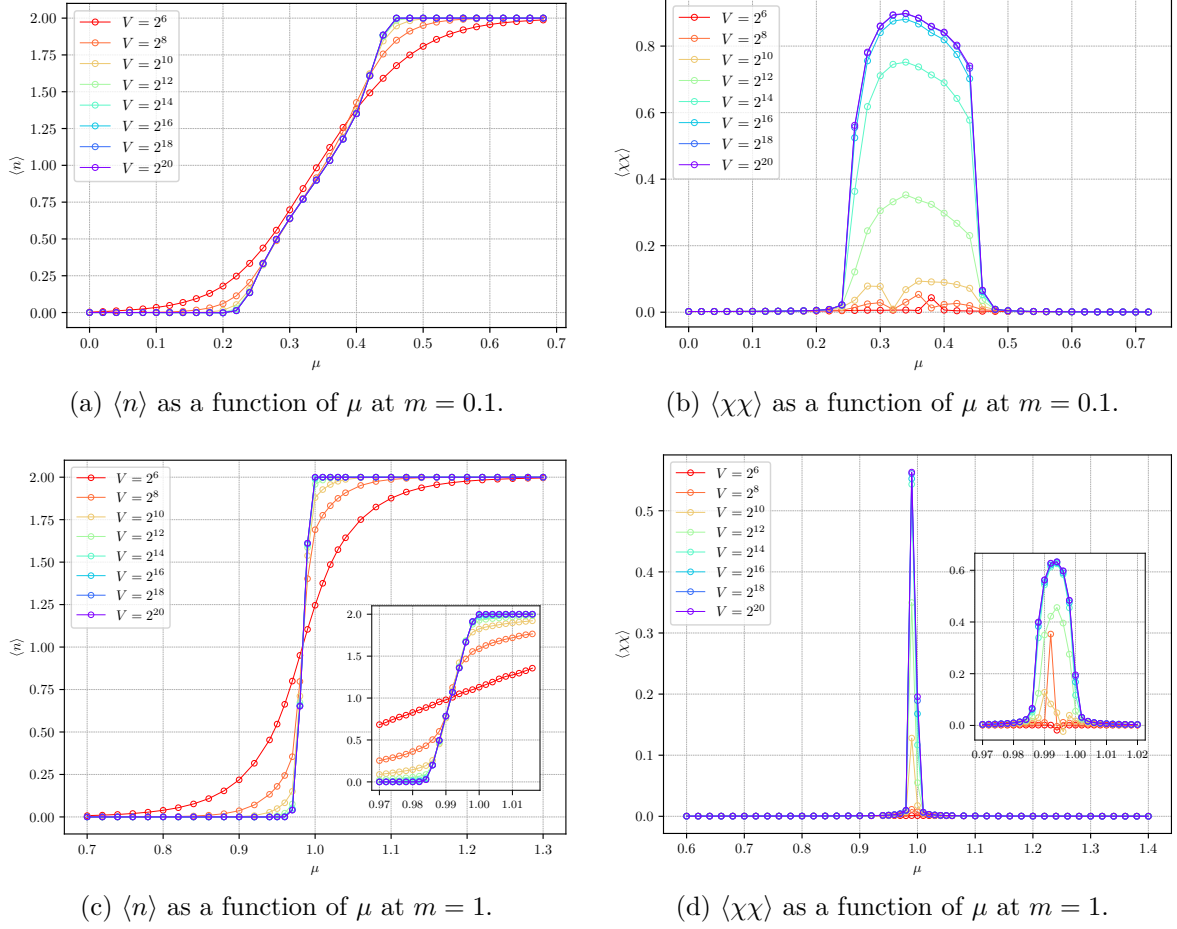


Figure 9. Volume dependence of physical quantities in the infinite coupling limit $\beta = 0$. The bond dimension in the calculations is $D = 84$. To evaluate the numerical differences in eqs. (4.3) and (4.7), we set $\Delta\mu = 0.04$ for $m = 0.1$, $\Delta\mu = 0.02$ for $m = 1$, and $\lambda = \Delta\lambda = 10^{-4}$. At $m = 1$, the insets show the volume dependence of $\langle n \rangle$ and $\langle \chi\chi \rangle$ in the intermediate phase, where $\langle n \rangle$ is evaluated by $\Delta\mu = 0.004$.

r	D'_1	D'_2	D'_3	D'_4	compression rate
1	224	224	224	224	100%
0.99999	86	86	84	84	2.07%
0.99995	68	68	66	66	0.800%
0.9999	61	61	59	59	0.514%
0.9995	46	46	43	43	0.155%
0.999	39	39	37	37	0.0827%
0.99	19	19	19	19	0.00518%

Table 2. Another example illustrating the efficiency of our initial tensor compression scheme with $m = 0.1$, $\beta = 0.8$, $\mu = 0.4$, $\lambda = 0$ and $K = 14$. As shown in figure 6, D'_1 and D'_2 denote the spatial bond dimensions and D'_3 and D'_4 denote the temporal ones. The compression rate in the last column is measured as the number of elements in $\mathcal{T}'_n$ divided by the number of elements in $\mathcal{T}_n$.

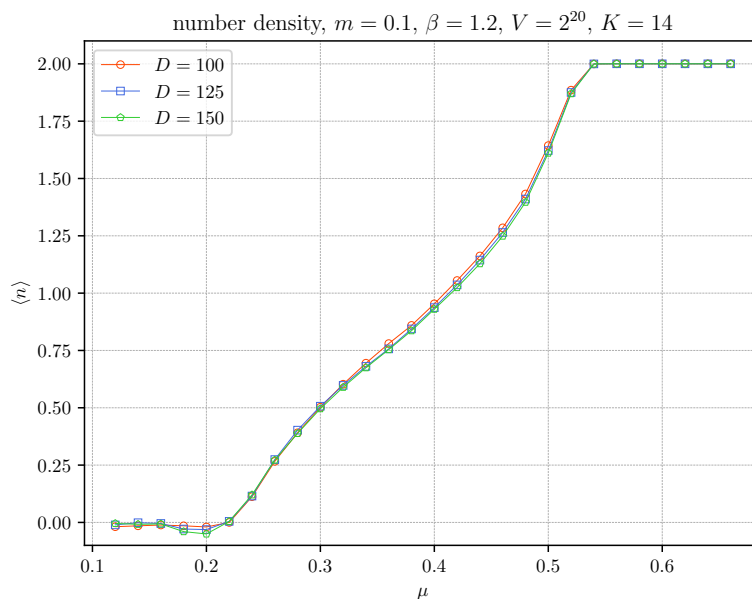


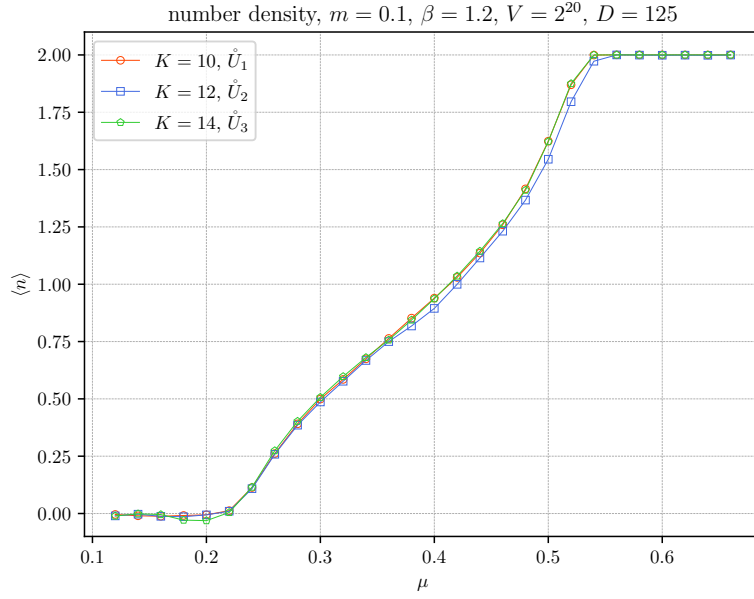
Figure 10. Quark number density $\langle n \rangle$ as a function of chemical potential μ at $m = 0.1$, $\beta = 1.2$, in the thermodynamic limit. We set the bond dimensions $D = 100, 125$, and 150 . The sample size for the discretization of gauge group integrations is $K = 14$. To evaluate the numerical difference in eq. (4.3), we set $\Delta\mu = 0.04$.

Next, we check the algorithmic parameter dependence of the BTRG results. At $m = 0.1$, $\beta = 1.2$ and $V = 2^{20}$, we calculate $\langle n \rangle$ as a function of μ using different bond dimension D as shown in figure 10, with a fixed number of sampled $SU(2)$ matrices $K = 14$. The qualitative behavior of $\langle n \rangle$ and the position of the two transition points $\mu_{c1/c2}$ are showing consistency as $D \geq 100$, despite some small deviations in the numerical values of $\langle n \rangle$.

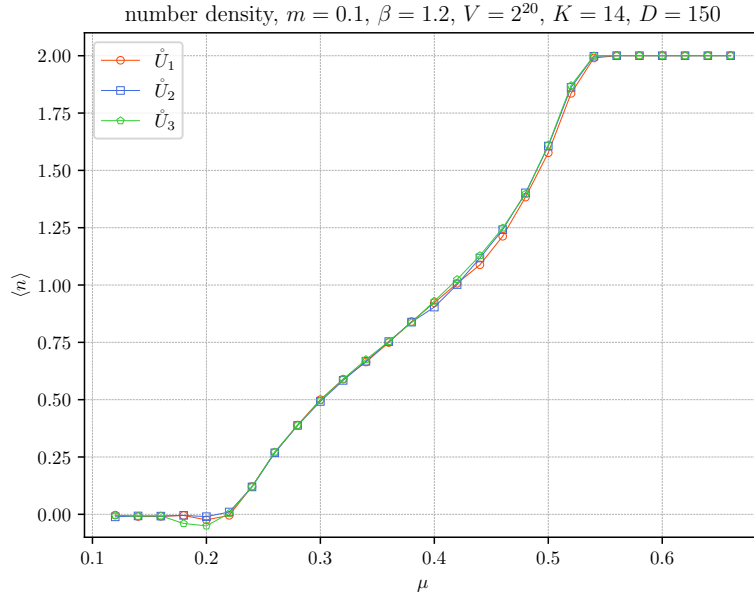
Similarly, $\langle n \rangle$ at $m = 0.1$, $\beta = 1.2$ and $V = 2^{20}$ calculated with different K and various matrix sets $\hat{U}_i$ are shown in figure 11(a) with $D = 125$. The results suggest that $K = 14$ suffices for our purpose and $\langle n \rangle$ obtained from different $\hat{U}_i$ exhibit similar qualitative behavior. Therefore, we always set $K = 14$ and $D = 150$ for the finite- β calculations in this study.

$\langle n \rangle$ at the same m , β and V , calculated with a fixed $K = 14$ and $D = 150$, using the three matrix sets adopted in figure 11(a), are illustrated in figure 11(b). Comparing figure 11(a) and 11(b), the results near μ_{c2} are affected more by the sample size K , rather than the choice of random matrix set. On the other hand, the results in the Silver-Blaze region rely more on which matrix set is being picked. Negative $\langle n \rangle$ is seen around $\mu = 0.2$, presumably because a finite number of $SU(2)$ matrices are used to approximate the gauge group integration. Since the $SU(2)$ matrices in each matrix set are randomly sampled, it is normal that some particular matrix set is able to give a better approximation under a fixed and finite sample size. For example, we can see from figure 11 that the negative $\langle n \rangle$ issue is eased when the matrix set $\hat{U}_2$ is picked.

We also comment on the practice of taking a statistical average over trials using distinct sets of $SU(2)$ matrices. It is believed to be able to reduce the systematic bias due to the



(a)



(b)

Figure 11. Quark number density $\langle n \rangle$ as a function of chemical potential μ at $m = 0.1$, $\beta = 1.2$, in the thermodynamic limit. We set $\Delta\mu = 0.04$ to evaluate the numerical difference in eq. (4.3). ((a)): the sample sizes for the discretization of gauge group integrations are set to be $K = 10, 12$, and 14 , with a distinct random matrix set $\hat{U}_i$ chosen for each K . The bond dimension is set to be $D = 125$. ((b)): the sample size is fixed to be $K = 14$, and the calculation is repeated using the same three matrix sets. The bond dimension is set to be $D = 150$.

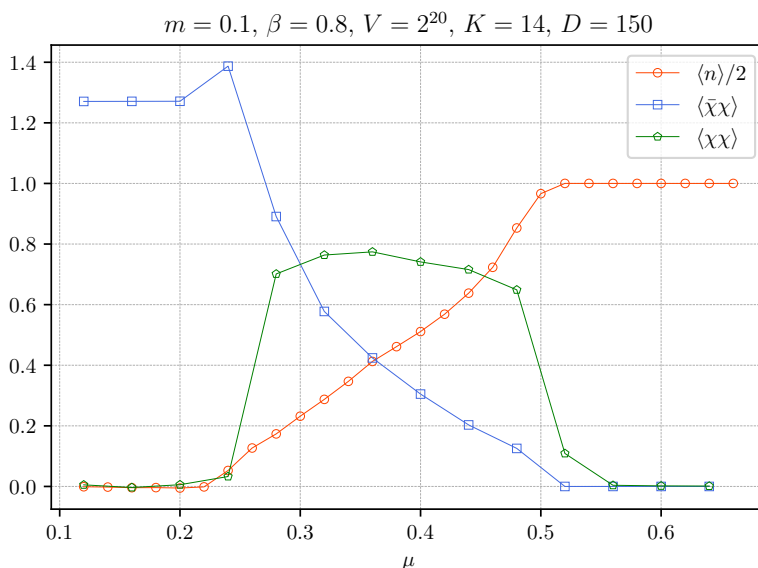


Figure 12. Quark number density $\langle n \rangle$, fermion condensate $\langle \bar{\chi}\chi \rangle$ and diquark condensate $\langle \chi\chi \rangle$ as a function of chemical potential μ at $m = 0.1$, $\beta = 0.8$, in the thermodynamic limit. The bond dimension in the calculations is $D = 150$. The sample size for the discretization of gauge group integrations is $K = 14$. To evaluate the numerical differences in eqs. (4.3), (4.4), and (4.7), we set $\Delta\mu = 0.04$, $\Delta m = 10^{-4}$, and $\lambda = \Delta\lambda = 10^{-4}$.

finite- K effect and improve the accuracy of the data such as reproducing the Silver-Blaze phenomenon better in the small μ regime. However, it is computationally demanding; we have found that so many trials are required to stabilize the numerical results, particularly the numerical differences. Therefore, we employ the same matrix set at different μ for a specific observable at a particular (m, β) .

Now, we see the numerical results of $\langle n \rangle$, $\langle \bar{\chi}\chi \rangle$ and $\langle \chi\chi \rangle$ as a function of μ in the thermodynamic limit $V = 2^{20}$, at a finite $\beta = 0.8$. For $m = 0.1$, the behavior of the three observables shown in figure 12 is similar to that at $\beta = 0$. We observe a more extended intermediate phase in μ as β becomes non-zero. At $m = 0.1$ and $\beta = 0.8$, $\mu_{c1} = 0.22$ and $\mu_{c2} = 0.52$. For $m = 1$, we see a sharp transition in figure 13 as in the infinite coupling limit while the intermediate phase shifts slightly in μ with $\mu_{c1} \approx 0.986$ and $\mu_{c2} \approx 1.004$. From the inset of figure 13, the crossover nature persists even when β becomes finite. The volume dependence of $\langle n \rangle$ and $\langle \chi\chi \rangle$ at $m = 0.1$ and $m = 1$ are shown in figure 14.

Finally, we describe the β dependence for μ_{c1} and μ_{c2} at $m = 0.1$. As shown in figure 15, the first transition point appears to be robust against β , i.e., $\mu_{c1} \approx 0.22$ for $0 \leq \beta \leq 1.6$ at $m = 0.1$. We expect that μ_{c1} becomes smaller as β increases because the mass gap vanishes in the limit of $\beta \rightarrow \infty$. To observe such behavior, we might need to compute the number density with $\beta > 1.6$ or to employ finer $\Delta\mu$ to improve the accuracy of the finite difference in eq. (4.3). On the other hand, the second transition point μ_{c2} is located at larger μ as β increases. It is expected because $\langle n \rangle$ does not saturate in regions of larger μ as the gauge coupling is weakened, namely approaching the continuum limit.

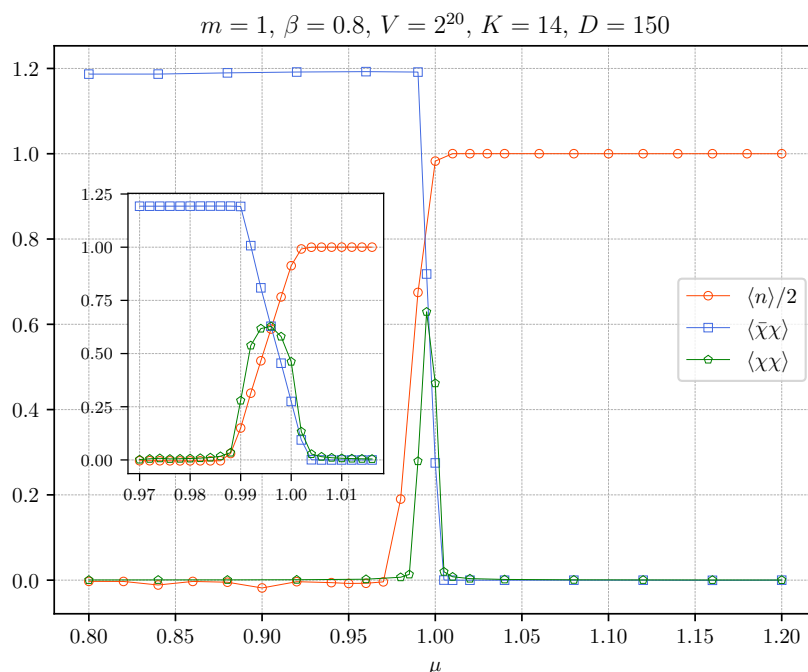


Figure 13. Quark number density $\langle n \rangle$, fermion condensate $\langle \bar{\chi}\chi \rangle$ and diquark condensate $\langle \chi\chi \rangle$ as a function of chemical potential μ at $m = 1, \beta = 0.8$, in the thermodynamic limit. The bond dimension in the calculations is $D = 150$. The sample size for the discretization of gauge group integrations is $K = 14$. To evaluate the numerical differences in eqs. (4.3), (4.4), and (4.7), we set $\Delta\mu = 0.02$, $\Delta m = 10^{-4}$, and $\lambda = \Delta\lambda = 10^{-4}$. The inset shows the three quantities in the intermediate phase, where $\langle n \rangle$ is evaluated using a finer $\Delta\mu = 0.004$.

5 Conclusion

In this work, we have investigated the $(1+1)$ -dimensional two-color QCD at finite density with staggered fermions using the TRG approach. We have used a random sampling method to discretize the gauge group integration and construct a Grassmann tensor network representation for the partition function. Since TRG calculations for non-Abelian gauge theories coupled with fermions are computationally challenging due to the very large initial bond dimension, we have proposed an efficient initial tensor compression scheme that can also be applied to other lattice models. We have evaluated the expectation values of the quark number density, fermion condensate, and diquark condensate. These observables are widely employed to investigate the phase structure of two-color QCD in higher dimensions. We have made BTRG calculations both at the infinite coupling limit and the finite coupling regime. Since there is no spontaneous breaking of continuous global symmetry in two dimensions, the fermion condensate and diquark condensate are computed by explicitly breaking the $U(1)_A$ and $U(1)_V$. Under this setting, we have found that the behavior of the observables calculated by the TRG approach is similar to that reported in a previous mean-field study [54] for the higher-dimensional two-color QCD. We have also studied how the positions of the two transition points vary as the gauge coupling changes.

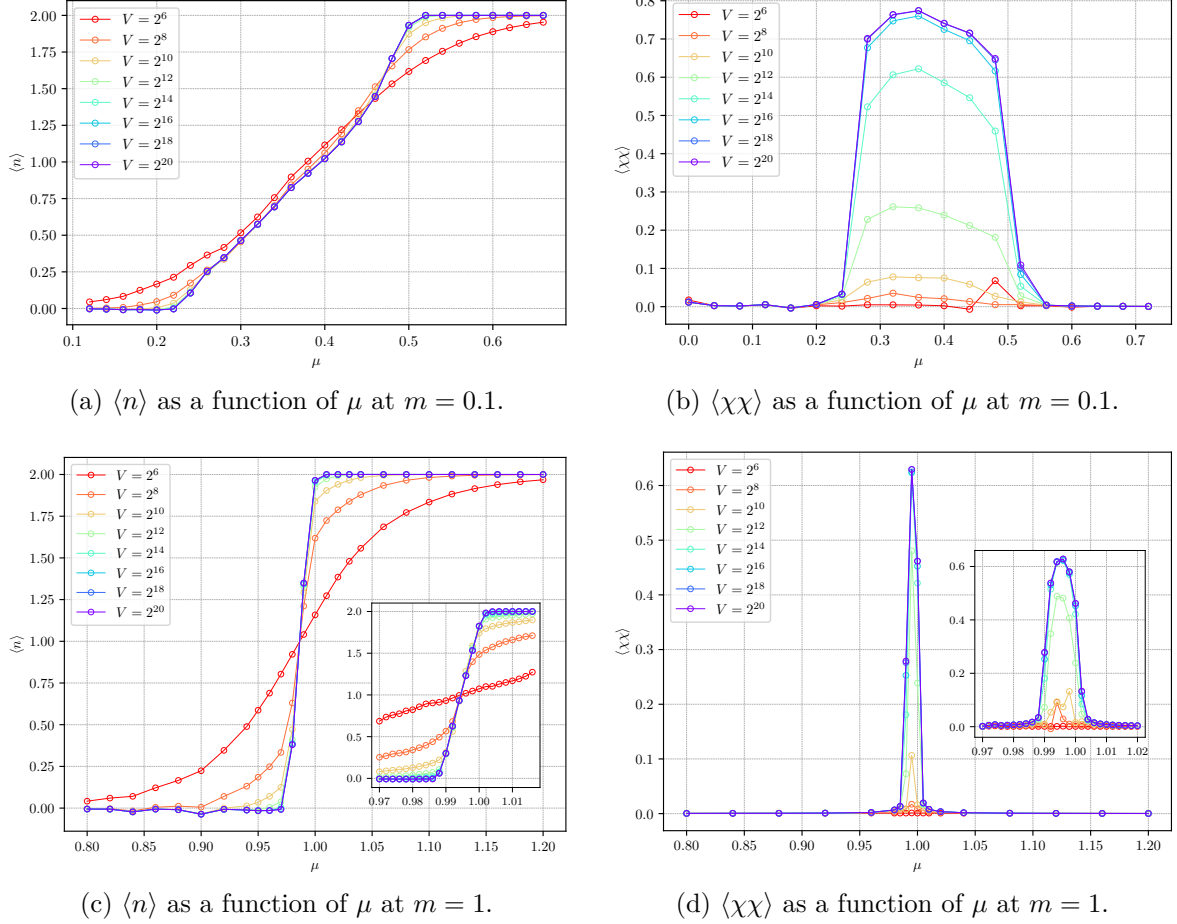


Figure 14. Volume dependence of physical quantities at $\beta = 0.8$. The bond dimension in the calculations is $D = 150$. The sample size for the discretization of gauge group integrations is $K = 14$. To evaluate the numerical differences in eqs. (4.3) and (4.7), we set $\Delta\mu = 0.04$ for $m = 0.1$, $\Delta\mu = 0.02$ for $m = 1$, and $\lambda = \Delta\lambda = 10^{-4}$. At $m = 1$, the insets show the volume dependence of $\langle n \rangle$ and $\langle \chi \chi \rangle$ in the intermediate phase, where $\langle n \rangle$ is evaluated by $\Delta\mu = 0.004$.

Our results encourage a future application of the TRG approach to the higher-dimensional two-color QCD. It is possible to improve our construction of initial tensors such that more $SU(2)$ matrices are used to discretize the gauge group integration, i.e., a larger K is allowed, without increasing the memory requirement significantly. In higher-dimensional cases where spontaneous breaking of continuous symmetry can exist, it is instructive to apply the TRG approach to evaluate the order parameters and perform extrapolations toward the chiral limit ($m = 0$) and the vanishing λ limit. We also emphasize that our Grassmann tensor network representation for the partition function can be extended to the three-color theory, which suffers from the sign problem at finite density, without conceptual difficulties.

Finally, we note that inhomogeneous phases in two-color QCD and related models [47, 66–68] have attracted attention recently. As another future direction, we will explore the applicability of the TRG approach in studying the spatial dependence of physical quantities.

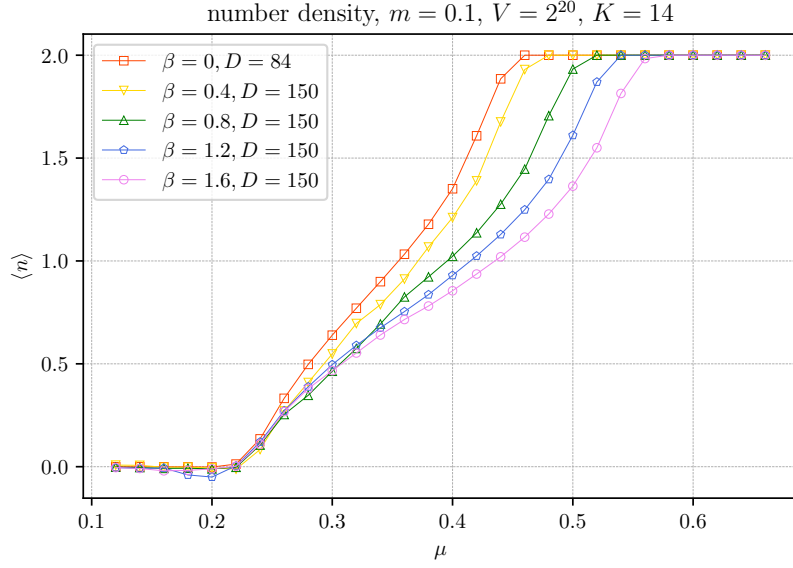


Figure 15. Quark number density $\langle n \rangle$ as a function of chemical potential μ at $m = 0.1$, $\beta = 0, 0.4, 0.8, 1.2, 1.6$ in the thermodynamic limit. The bond dimension in the infinite coupling calculation is $D = 84$, and the bond dimension in the finite β calculations is $D = 150$. At a finite β , the sample size for the discretization of gauge group integrations is $K = 14$. To evaluate the numerical differences in eq. (4.3), we set $\Delta\mu = 0.04$.

Acknowledgments

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A Coefficients of the Grassmann tensor $\mathcal{F}$

Here, we discuss one method to derive the tensor elements of F in eq. (2.10). For simplicity, we consider the two-color case where $N = 2$. We also label the spacetime direction ν by $\nu = x, t$ not by $\nu = 1, 2$. In this case, the integration over the original staggered fermions is written as

$$\begin{aligned} \mathcal{F} &= \int d\chi_1 d\bar{\chi}_1 d\chi_2 d\bar{\chi}_2 e^{-m(\bar{\chi}_1\chi_1 + \bar{\chi}_2\chi_2)} (1 + \bar{\chi}_1 A) (1 + \chi_1 B) (1 + \bar{\chi}_2 C) (1 + \chi_2 D) \\ &= ABCD + mCD + mAB + m^2, \end{aligned} \quad (\text{A.1})$$

where A , B , C , and D are sums of terms with one auxiliary Grassmann variable, and their expressions are given by

$$A = -\eta_{x,1} - \eta_{t,1} + (U_x^\dagger)_{11}\bar{\zeta}_{x,1} + (U_x^\dagger)_{12}\bar{\zeta}_{x,2} + (U_t^\dagger)_{11}\bar{\zeta}_{t,1} + (U_t^\dagger)_{12}\bar{\zeta}_{t,2}, \quad (\text{A.2})$$

$$B = \frac{1}{2} \left[-(U_x)_{11}\bar{\eta}_{x,1} - (U_x)_{21}\bar{\eta}_{x,2} - a_+(U_t)_{11}\bar{\eta}_{t,1} - a_+(U_t)_{21}\bar{\eta}_{t,2} + \zeta_{x,1} + a_-\zeta_{t,1} \right], \quad (\text{A.3})$$

$$C = -\eta_{x,2} - \eta_{t,2} + (U_x^\dagger)_{21}\bar{\zeta}_{x,1} + (U_x^\dagger)_{22}\bar{\zeta}_{x,2} + (U_t^\dagger)_{21}\bar{\zeta}_{t,1} + (U_t^\dagger)_{22}\bar{\zeta}_{t,2}, \quad (\text{A.4})$$

$$D = \frac{1}{2} \left[- (U_x)_{12}\bar{\eta}_{x,1} - (U_x)_{22}\bar{\eta}_{x,2} - a_+(U_t)_{12}\bar{\eta}_{t,1} - a_+(U_t)_{22}\bar{\eta}_{t,2} + \zeta_{x,2} + a_-\zeta_{t,2} \right], \quad (\text{A.5})$$

with $a_\pm = e^{\pm\mu} p_t(n)$. After expanding out the second line in eq. (A.1), one can compare like terms in eqs. (A.1) and (2.10), then obtain the elements of coefficient tensor F . When a diquark source term $\sum_n \lambda [\chi^T(n)\sigma_2\chi(n) + \bar{\chi}(n)\sigma_2\bar{\chi}^T(n)]/2$ is added to the action, then the Grassmann tensor $\mathcal{F}$ in eq. (2.10) is given by $ABCD + m(AB + CD) + i\lambda(AC + BD) + m^2 + \lambda^2$.

B Construction of ρ

In section 3.2, we introduced a Hermitian matrix as $\rho_B = M_B^\dagger M_B$. It is computationally demanding to obtain ρ_B directly through this formula since it is a multiplication between a $16K \times 16^3K^3$ matrix and a $16^3K^3 \times 16K$ matrix.

In this study, we obtain ρ_B (and ρ_A by similar steps) following another equivalent method. First, we perform the following SVD on the coefficient tensor T_n as

$$(-1)^{f_x(f_t+f_{x'})+f_{x'}f_t}(T_n)_{xtx't'} = \sum_{y=1}^{16^2K^2} (U_B)_{x'ty}(s_B)_y(V_B^\dagger)_{yxt'}. \quad (\text{B.1})$$

Since the direct calculation of eq. (B.1) is also demanding, we employ a randomized SVD [5] with the oversampling parameter $p \sim 0.07(16K)^2$ and the iteration number of QR decomposition $r' = 7$ in this step.

Then we substitute eq. (B.1) into the definition of ρ_B :

$$\begin{aligned} (\rho_B)_{x\tilde{x}} &= \sum_{t,x',t'} (M_B^*)_{(tx't')x} (M_B)_{(tx't')\tilde{x}} \\ &= \sum_{t,x',t'} (-1)^{f_x(f_t+f_{x'})+f_{x'}f_t} (T_n^*)_{xtx't'} (-1)^{f_{\tilde{x}}(f_t+f_{x'})+f_{x'}f_t} (T_n)_{\tilde{x}tx't'} \\ &= \sum_{t,x',t'} \sum_{y,\tilde{y}} (U_B)_{x'ty}^* (s_B)_y (V_B^\dagger)_{yxt'}^* (U_B)_{x't\tilde{y}} (s_B)_{\tilde{y}} (V_B^\dagger)_{\tilde{y}\tilde{x}t'} \\ &= \sum_{t'} \sum_{y,\tilde{y}} \delta_{y,\tilde{y}} (s_B)_y (V_B^\dagger)_{yxt'}^* (s_B)_{\tilde{y}} (V_B^\dagger)_{\tilde{y}\tilde{x}t'} \\ &= \sum_{t'} \sum_y (s_B)_y^2 (V_B^\dagger)_{yxt'}^* (V_B^\dagger)_{y\tilde{x}t'}. \end{aligned} \quad (\text{B.2})$$

From the third to the fourth line of eq. (B.2), we used the property $U_B^\dagger U_B = I$ of an SVD. eq. (B.2) shows that ρ_B can be made once s_B and $V_B^\dagger$ in eq. (B.1) are obtained.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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