

Impurity tensor of the pseudoscalar condensate in the GN model

Consider the single-flavor Gross-Neveu(GN) model on the two dimensional space-time lattice, the action S looks like:

$$S = -\frac{1}{2} \sum_{n \in \Lambda_2} \sum_{\nu=1,2} [e^{\mu\delta_{\nu,2}} \bar{\psi}(n) (rI - \gamma_{\nu}) \psi(n + \hat{\nu}) + e^{-\mu\delta_{\nu,2}} \bar{\psi}(n + \hat{\nu}) (rI + \gamma_{\nu}) \psi(n)] \\ + (m + 2r) \sum_n \bar{\psi}(n) \psi(n) - \frac{g^2}{2} \sum_n [(\bar{\psi}(n) \psi(n))^2 + (\bar{\psi}(n) i\gamma_5 \psi(n))^2] \quad (1)$$

$\psi = (\psi_1, \psi_2)$ denotes the two-component Grassmann-valued field. The m is the mass parameter, the r is the Wilson parameter which is set as 1, and $g_{\sigma}^2 = g_{\pi}^2 = g^2$ denote the strength of the four-Fermi interaction. I is the 2×2 identity matrix. In the chiral representaion, the gamma matrices are set as $\gamma_1 \equiv \hat{\sigma}_x$, $\gamma_2 \equiv \hat{\sigma}_y$ and $\gamma_5 \equiv \hat{\sigma}_z$. The Grassmann tensor of model (1) generally takes the form:

$$T_{(\eta_1 \xi_1)(\eta_2 \xi_2)(\bar{\xi}_1 \bar{\eta}_1)(\bar{\xi}_2 \bar{\eta}_2)} = \sum_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)} T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)} (\eta_1^{i_1} \xi_1^{j_1})(\eta_2^{i_2} \xi_2^{j_2})(\bar{\xi}_1^{j'_1} \bar{\eta}_1^{i'_1})(\bar{\xi}_2^{j'_2} \bar{\eta}_2^{i'_2}) \quad (2)$$

Now we try to derive the concrete form of $T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)}$ based on the general procedure. First, the hopping matrices X^{ν} and Y^{ν} are determined and decomposed as :

$$X^{\nu=1} = -\frac{1}{2} (rI - \hat{\sigma}_x) = -\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \equiv -U^{X^1} \sigma^{X^1} V^{X^1 \dagger} \quad (3)$$

$$X^{\nu=2} = -\frac{1}{2} e^{\mu} (rI - \hat{\sigma}_y) = -\frac{e^{\mu}}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \equiv -U^{X^2} \sigma^{X^2} V^{X^2 \dagger} \quad (4)$$

$$Y^{\nu=1} = -\frac{1}{2} (rI + \hat{\sigma}_x) = -\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \equiv -U^{Y^1} \sigma^{Y^1} V^{Y^1 \dagger} \quad (5)$$

$$Y^{\nu=2} = -\frac{1}{2} e^{-\mu} (rI + \hat{\sigma}_y) = -\frac{e^{-\mu}}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \equiv -U^{Y^2} \sigma^{Y^2} V^{Y^2 \dagger} \quad (6)$$

The matrices from (3) to (6) are:

$$U^{X^1} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad \sigma^{X^1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} \quad V^{X^1 \dagger} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned}
U^{X^2} &= \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} & \sigma^{X^2} &= \begin{pmatrix} e^\mu/2 & 0 \\ 0 & 0 \end{pmatrix} & V^{X^2\dagger} &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \\
U^{Y^1} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \sigma^{Y^1} &= \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} & V^{Y^1\dagger} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
U^{Y^2} &= \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} & \sigma^{Y^2} &= \begin{pmatrix} e^{-\mu}/2 & 0 \\ 0 & 0 \end{pmatrix} & V^{Y^2\dagger} &= \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}
\end{aligned}$$

Notice only the (1,1) component of all the singular matrices σ is non-zero. We substitute the decompositions above into (1), and introduce new fields $\{\bar{\chi}, \chi\}$ through performing unitary transformations to the original fields $\{\bar{\psi}, \psi\}$, we arrive at the following exponential terms appeared in e^{-S} :

$$e^{\sigma_1^{X^1} \bar{\chi}_1^{X^1}(n) \chi_1^{X^1}(n+\hat{1})} e^{\sigma_1^{X^2} \bar{\chi}_1^{X^2}(n) \chi_1^{X^2}(n+\hat{2})} \quad (7)$$

$$e^{\sigma_1^{Y^1} \bar{\chi}_1^{Y^1}(n+\hat{1}) \chi_1^{Y^1}(n)} e^{\sigma_1^{Y^2} \bar{\chi}_1^{Y^2}(n+\hat{2}) \chi_1^{Y^2}(n)} \quad (8)$$

We introduce a pair of auxiliary bond Grassmann numbers to each hopping term:

$$e^{\sigma_1^{X^1} \bar{\chi}_1^{X^1}(n) \chi_1^{X^1}(n+\hat{1})} = \int_{\bar{\eta}_1(n) \eta_1(n)} e^{\sqrt{\sigma_1^{X^1}} \bar{\chi}_1^{X^1}(n) \eta_1(n)} e^{\sqrt{\sigma_1^{X^1}} \bar{\eta}_1(n) \chi_1^{X^1}(n+\hat{1})} \quad (9)$$

$$e^{\sigma_1^{X^2} \bar{\chi}_1^{X^2}(n) \chi_1^{X^2}(n+\hat{2})} = \int_{\bar{\eta}_2(n) \eta_2(n)} e^{\sqrt{\sigma_1^{X^2}} \bar{\chi}_1^{X^2}(n) \eta_2(n)} e^{\sqrt{\sigma_1^{X^2}} \bar{\eta}_2(n) \chi_1^{X^2}(n+\hat{2})} \quad (10)$$

$$e^{\sigma_1^{Y^1} \bar{\chi}_1^{Y^1}(n+\hat{1}) \chi_1^{Y^1}(n)} = \int_{\bar{\xi}_1(n) \xi_1(n)} e^{\sqrt{\sigma_1^{Y^1}} \bar{\chi}_1^{Y^1}(n+\hat{1}) \bar{\xi}_1(n)} e^{-\sqrt{\sigma_1^{Y^1}} \xi_1(n) \chi_1^{Y^1}(n)} \quad (11)$$

$$e^{\sigma_1^{Y^2} \bar{\chi}_1^{Y^2}(n+\hat{2}) \chi_1^{Y^2}(n)} = \int_{\bar{\xi}_2(n) \xi_2(n)} e^{\sqrt{\sigma_1^{Y^2}} \bar{\chi}_1^{Y^2}(n+\hat{2}) \bar{\xi}_2(n)} e^{-\sqrt{\sigma_1^{Y^2}} \xi_2(n) \chi_1^{Y^2}(n)} \quad (12)$$

We select terms containing original fields on the n -th site and neglect the n -th labels hereafter. Expand the (9) and substitute the matrix elements of the matrices:

$$\begin{aligned}
e^{\sqrt{\sigma_1^{X^1}} \bar{\chi}_1^{X^1} \eta_1(n)} &= e^{\sqrt{\sigma_1^{X^1}} (U_{11}^{X^1} \bar{\psi}_1 + U_{21}^{X^1} \bar{\psi}_2) \eta_1(n)} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{i_1} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} \eta_1^{i_1}(n)
\end{aligned} \quad (13)$$

$$\begin{aligned}
e^{\sqrt{\sigma_1^{X^1}} \bar{\eta}_1(n-\hat{1}) \chi_1^{X^1}} &= e^{\sqrt{\sigma_1^{X^1}} \bar{\eta}_1(n-\hat{1}) (V_{11}^{\dagger X^1} \psi_1 + V_{12}^{\dagger X^1} \psi_2)} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{i'_1} \bar{\eta}_1^{i'_1}(n-\hat{1}) (\psi_1 - \psi_2)^{i'_1}
\end{aligned} \quad (14)$$

Expand the (10):

$$\begin{aligned}
e^{\sqrt{\sigma_1^{X^2}} \bar{\chi}_1^{X^2} \eta_2(n)} &= e^{\sqrt{\sigma_1^{X^2}} (U_{11}^{X^2} \bar{\psi}_1 + U_{21}^{X^2} \bar{\psi}_2) \eta_2(n)} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{i_2} (e^{\frac{\mu}{2}})^{i_2} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} \eta_2^{i_2}(n)
\end{aligned} \tag{15}$$

$$\begin{aligned}
e^{\sqrt{\sigma_1^{X^2}} \bar{\eta}_2(n-\hat{2}) \chi_1^{X^2}} &= e^{\sqrt{\sigma_1^{X^2}} \bar{\eta}_2(n-\hat{2}) (V_{11}^{\dagger X^2} \psi_1 + V_{12}^{\dagger X^2} \psi_2)} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{i'_2} (e^{\frac{\mu}{2}})^{i'_2} \bar{\eta}_2^{i'_2}(n-\hat{2}) (\psi_1 + i\psi_2)^{i'_2}
\end{aligned} \tag{16}$$

Expand the (11):

$$\begin{aligned}
e^{\sqrt{\sigma_1^{Y^1}} \bar{\chi}_1^{Y^1} \xi_1(n-\hat{1})} &= e^{\sqrt{\sigma_1^{Y^1}} (U_{11}^{Y^1} \bar{\psi}_1 + U_{21}^{Y^1} \bar{\psi}_2) \xi_1(n-\hat{1})} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{j'_1} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} \bar{\xi}_1^{j'_1}(n-\hat{1})
\end{aligned} \tag{17}$$

$$\begin{aligned}
e^{-\sqrt{\sigma_1^{Y^1}} \xi_1(n) \chi_1^{Y^1}} &= e^{-\sqrt{\sigma_1^{Y^1}} \xi_1(n) (V_{11}^{\dagger Y^1} \psi_1 + V_{12}^{\dagger Y^1} \psi_2)} \\
&= (-1)^{j_1} \left(\frac{\sqrt{2}}{2}\right)^{j_1} \xi_1^{j_1}(n) (\psi_1 + \psi_2)^{j_1}
\end{aligned} \tag{18}$$

Expand the (12):

$$\begin{aligned}
e^{\sqrt{\sigma_1^{Y^2}} \bar{\chi}_1^{Y^2} \bar{\xi}_2(n-\hat{2})} &= e^{\sqrt{\sigma_1^{Y^2}} (U_{11}^{Y^2} \bar{\psi}_1 + U_{21}^{Y^2} \bar{\psi}_2) \bar{\xi}_2(n-\hat{2})} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{j'_2} (e^{-\frac{\mu}{2}})^{j'_2} (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} \bar{\xi}_2^{j'_2}(n-\hat{2})
\end{aligned} \tag{19}$$

$$\begin{aligned}
e^{-\sqrt{\sigma_1^{Y^2}} \xi_2(n) \chi_1^{Y^2}} &= e^{-\sqrt{\sigma_1^{Y^2}} \xi_2(n) (V_{11}^{\dagger Y^2} \psi_1 + V_{12}^{\dagger Y^2} \psi_2)} \\
&= (-1)^{j_2} \left(\frac{\sqrt{2}}{2}\right)^{j_2} (e^{-\frac{\mu}{2}})^{j_2} \xi_2^{j_2}(n) (\psi_1 - i\psi_2)^{j_2}
\end{aligned} \tag{20}$$

Expand the rest terms in (1):

$$e^{-(m+2r)\bar{\psi}\psi} = (1 - (m+2r)\bar{\psi}\psi + (m+2r)^2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2) \tag{21}$$

$$e^{\frac{g^2}{2} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2]} = 1 + 2g^2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \quad (22)$$

Starting from (22) to (13), we move all the terms containing original fields $\psi_1, \psi_2, \bar{\psi}_1$ and $\bar{\psi}_2$ to the left side. This process would generate a sign factor of $(-1)^{j_2+j_1+i'_2+i'_1}$, which can be incorporated with that in (18) and (20), leading to $(-1)^{i'_2+i'_1}$. We then adjust the order of the original and new Grassmann fields at the same time while maintaining the mirror symmetry within them (thus no additional sign factor appears), and arrives at:

$$\begin{aligned} & (1 - (m + 2r)\bar{\psi}\psi + (m + 2r)^2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2) (1 + 2g^2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2) \\ & \times (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} \\ & \times (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1} \\ & \times \xi_1^{j_1}(n) \xi_2^{j_2}(n) \bar{\eta}_1^{i'_1}(n - \hat{1}) \bar{\eta}_2^{i'_2}(n - \hat{2}) \eta_1^{i_1}(n) \eta_2^{i_2}(n) \bar{\xi}_1^{j'_1}(n - \hat{1}) \bar{\xi}_2^{j'_2}(n - \hat{2}) \end{aligned} \quad (23)$$

Now we arrange the new Grassmann fields into the desired order (move from the underlined position to the blue-star position sequentially):

$$\begin{aligned} & (*) \xi_1^{j_1}(n) \xi_2^{j_2}(n) \bar{\eta}_1^{i'_1}(n - \hat{1}) \bar{\eta}_2^{i'_2}(n - \hat{2}) \underline{\eta_1^{i_1}(n) \eta_2^{i_2}(n)} \bar{\xi}_1^{j'_1}(n - \hat{1}) \bar{\xi}_2^{j'_2}(n - \hat{2}) \\ \rightarrow & (\eta_1^{i_1}(n) \xi_1^{j_1}(n)) (*) \xi_2^{j_2}(n) \bar{\eta}_1^{i'_1}(n - \hat{1}) \bar{\eta}_2^{i'_2}(n - \hat{2}) \underline{\eta_2^{i_2}(n) \bar{\xi}_1^{j'_1}(n - \hat{1}) \bar{\xi}_2^{j'_2}(n - \hat{2})} \\ \rightarrow & (\eta_1^{i_1}(n) \xi_1^{j_1}(n)) (\eta_2^{i_2}(n) \xi_2^{j_2}(n)) (*) \bar{\eta}_1^{i'_1}(n - \hat{1}) \bar{\eta}_2^{i'_2}(n - \hat{2}) \underline{\bar{\xi}_1^{j'_1}(n - \hat{1}) \bar{\xi}_2^{j'_2}(n - \hat{2})} \\ \rightarrow & (\eta_1^{i_1}(n) \xi_1^{j_1}(n)) (\eta_2^{i_2}(n) \xi_2^{j_2}(n)) \left(\bar{\xi}_1^{j'_1}(n - \hat{1}) \bar{\eta}_1^{i'_1}(n - \hat{1}) \right) (*) \bar{\eta}_2^{i'_2}(n - \hat{2}) \underline{\bar{\xi}_2^{j'_2}(n - \hat{2})} \\ \rightarrow & (\eta_1^{i_1}(n) \xi_1^{j_1}(n)) (\eta_2^{i_2}(n) \xi_2^{j_2}(n)) \left(\bar{\xi}_1^{j'_1}(n - \hat{1}) \bar{\eta}_1^{i'_1}(n - \hat{1}) \right) \left(\bar{\xi}_2^{j'_2}(n - \hat{2}) \bar{\eta}_2^{i'_2}(n - \hat{2}) \right) \end{aligned}$$

The accumulated sign factors in the above process are:

$$(-1)^{i_1 \times (i'_2 + i'_1 + j_2 + j_1) + i_2 \times (i'_2 + i'_1 + j_2) + j'_1 \times (i'_2 + i'_1) + j'_2 \times i'_2} \quad (24)$$

Now we would like to perform the Grassmann integral $\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2$ to integrate the original Grassmann fields on the n -th site. (23) contain the following contributions:

$$(2g^2 + (m + 2r)^2) \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \rightarrow (2g^2 + (m + 2r)^2) \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (25)$$

$$-(m+2r)\bar{\psi}_1\psi_1 \rightarrow -(-1)^{i_1+i_2+i'_1+j_2} \times (i)^{i_2+i'_2+j_2+j'_2} \times (m+2r)\delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} \quad (26)$$

$$-(m+2r)\bar{\psi}_2\psi_2 \rightarrow -(m+2r)\delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} \quad (27)$$

$$1 \rightarrow -\bar{A}_{i_1,i_2,j'_1,j'_2}A_{j_1,j_2,i'_1,i'_2} \quad (28)$$

The minus sign in (28) comes from adjusting the Grassmann integral measure from $\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2$ to $\int d\psi_2 d\psi_1 d\bar{\psi}_2 d\bar{\psi}_1$. The $\bar{A}_{i_1,i_2,j'_1,j'_2}$ is given by the following Grassmann integral:

$$\bar{A}_{i_1,i_2,j'_1,j'_2} = \int d\bar{\psi}_2 d\bar{\psi}_1 (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} \quad (29)$$

Similarly, the A_{j_1,j_2,i'_1,i'_2} is given by the following Grassmann integral:

$$A_{j_1,j_2,i'_1,i'_2} = \int d\psi_2 d\psi_1 (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1} \quad (30)$$

The non-zero tensor elements of $\bar{A}_{i_1,i_2,j'_1,j'_2}$ and A_{j_1,j_2,i'_1,i'_2} are listed:

$$\begin{aligned} \bar{A}_{1,1,0,0} &= -1 + i, \quad \bar{A}_{1,0,1,0} = -2, \quad \bar{A}_{1,0,0,1} = -1 - i, \quad \bar{A}_{0,1,1,0} = -1 - i, \quad \bar{A}_{0,1,0,1} = -2i, \quad \bar{A}_{0,0,1,1} = 1 - i \\ A_{1,1,0,0} &= 1 + i, \quad A_{1,0,1,0} = 2, \quad A_{1,0,0,1} = 1 - i, \quad A_{0,1,1,0} = 1 - i, \quad A_{0,1,0,1} = -2I, \quad A_{0,0,1,1} = -1 - i \end{aligned}$$

Finally, we gather all the information below and obtain the coefficient tensor $T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)}$:

$$\begin{aligned} T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)} &= (-1)^{i_1 \times (i'_2 + i'_1 + j_2 + j_1) + i_2 \times (i'_2 + i'_1 + j_2) + j'_1 \times (i'_2 + i'_1) + j'_2 \times i'_2 + i'_1 + i'_2} \times e^{\frac{\mu}{2}(i_2 + i'_2 - j_2 - j'_2)} \\ &\times \left(\frac{\sqrt{2}}{2} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \times [(2g^2 + (m+2r)^2) \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \\ &- (-1)^{i_1+i_2+i'_1+j_2} \times (+i)^{i_2+i'_2+j_2+j'_2} \times (m+2r) \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \\ &- (m+2r) \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} - \bar{A}_{i_1,i_2,j'_1,j'_2} A_{j_1,j_2,i'_1,i'_2}] \end{aligned}$$

The expectation value of the pseudoscalar condensate $\langle \bar{\psi} i \gamma_5 \psi \rangle$:

$$\langle \bar{\psi} i \gamma_5 \psi \rangle = \int \frac{\bar{\psi} i \gamma_5 \psi e^{-S}}{Z} = \int \frac{i (\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2) e^{-S}}{Z} \quad (31)$$

Therefore, we only need to add the term $i (\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)$ to (23), and perform the Grassmann integral $\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2$. The contributions are:

$$i \bar{\psi}_1 \psi_1 \rightarrow -i(m + 2r) \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (32)$$

$$i \bar{\psi}_1 \psi_1 \rightarrow i(+i)^{j'_2+i_2+i'_2+j_2} (-1)^{i_2+i_1+i'_1+j_2} \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \quad (33)$$

$$-i \bar{\psi}_2 \psi_2 \rightarrow i(m + 2r) \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (34)$$

$$-i \bar{\psi}_2 \psi_2 \rightarrow -i \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \quad (35)$$

Therefore, the impurity tensor corresponds to the pseudoscalar condensate $\langle \bar{\psi} i \gamma_5 \psi \rangle$ can be obtained as:

$$\begin{aligned} T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)}^{\bar{\psi} i \gamma_5 \psi} &= (-1)^{i_1 \times (i'_2+i'_1+j_2+j_1) + i_2 \times (i'_2+i'_1+j_2) + j'_1 \times (i'_2+i'_1) + j'_2 \times i'_2+i'_1+i'_2} \times e^{\frac{\mu}{2}(i_2+i'_2-j_2-j'_2)} \\ &\times \left(\frac{\sqrt{2}}{2} \right)^{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2} \times \left[i(+i)^{j'_2+i_2+i'_2+j_2} (-1)^{i_2+i_1+i'_1+j_2} \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \right. \\ &\left. - i \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \right] \end{aligned}$$

The pseudoscalar condensate $\langle \bar{\psi} i \gamma_5 \psi \rangle$ can be also computed through finite difference methods by adding the following term in the action (1):

$$S_h = -h \sum_n \bar{\psi}(n) i \gamma_5 \psi(n) \quad (36)$$

Similarly, it would contribute a term $e^{ih(\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)}$ to (23), which would expand like:

$$e^{ih(\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)} = 1 + ih (\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2) + h^2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \quad (37)$$

The additional contributions from (37) are:

$$ih \bar{\psi}_1 \psi_1 \rightarrow -i(m + 2r)h \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (38)$$

$$ih \bar{\psi}_1 \psi_1 \rightarrow (-1)^{i_1+i_2+i'_1+j_2} \times (i)^{i_2+i'_2+j_2+j'_2} \times ih \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \quad (39)$$

$$-ih \bar{\psi}_2 \psi_2 \rightarrow i(m + 2r)h \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (40)$$

$$-ih \bar{\psi}_2 \psi_2 \rightarrow -ih \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \quad (41)$$

$$h^2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \rightarrow h^2 \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (42)$$

Therefore, the Grassmann tensor now looks like:

$$\begin{aligned} T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)} &= (-1)^{i_1 \times (i'_2+i'_1+j_2+j_1) + i_2 \times (i'_2+i'_1+j_2) + j'_1 \times (i'_2+i'_1) + j'_2 \times i'_2 + i'_1 + i'_2} \times e^{\frac{h}{2}(i_2+i'_2-j_2-j'_2)} \\ &= \left(\frac{\sqrt{2}}{2} \right)^{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2} \times [(2g^2 + (m + 2r)^2 + h^2) \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \\ &- (-1)^{i_1+i_2+i'_1+j_2} \times (+i)^{i_2+i'_2+j_2+j'_2} \times (m + 2r - ih) \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \\ &- (m + 2r + ih) \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} - \bar{A}_{i_1,i_2,j'_1,j'_2} A_{j_1,j_2,i'_1,i'_2}] \end{aligned}$$

现在我们考虑一个更 general 的情况，也就是对存在 external field (36) 的模型使用杂志点方法。存在 external field 展开后的 action 是：

$$\begin{aligned}
& (1 - (m + 2r)\bar{\psi}\psi + (m + 2r)^2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2) (1 + 2g^2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2) \\
& \times [1 + ih(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) + h^2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2] \\
& \times (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i'_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i'_1} \\
& \times (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1}
\end{aligned} \tag{43}$$

如果在 (43) 的基础上，再添加上杂志点的贡献：

$$i(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \tag{44}$$

则执行 Grassmann 积分后得到的非零项为：

$$\begin{aligned}
i\bar{\psi}_1\psi_1 & \rightarrow -i(m + 2r)\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \\
i\bar{\psi}_1\psi_1 & \rightarrow h\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \\
i\bar{\psi}_1\psi_1 & \rightarrow i(-1)^{i_2+i_1+i'_1+j_2}(i)^{j'_2+i_2+i'_2+j_2}\delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} \\
-i\bar{\psi}_2\psi_2 & \rightarrow i(m + 2r)\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \\
-i\bar{\psi}_2\psi_2 & \rightarrow h\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \\
-i\bar{\psi}_2\psi_2 & \rightarrow -i\delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1}
\end{aligned}$$

最终的杂志点张量因而可以表示为：

$$\begin{aligned}
& T_{(i_1j_1)(i_2j_2)(i'_1j'_1)(i'_2j'_2)}^{\bar{\psi}\gamma_5\psi} = (-1)^{i_1 \times (i'_2+i'_1+j_2+j_1) + i_2 \times (i'_2+i'_1+j_2) + j'_1 \times (i'_2+i'_1) + j'_2 \times i'_2 + i'_1 + i'_2} \times e^{\frac{\mu}{2}(i_2+i'_2-j_2-j'_2)} \\
& \times \left(\frac{\sqrt{2}}{2} \right)^{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2} \times \left[i(+i)^{j'_2+i_2+i'_2+j_2} (-1)^{i_2+i_1+i'_1+j_2} \delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} \right. \\
& \left. - i\delta_{i_1+i_2+j'_1+j'_2,1} \delta_{j_1+j_2+i'_1+i'_2,1} + 2h\delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \right]
\end{aligned}$$

再考虑 $\langle (\bar{\psi} i \gamma_5 \psi)^2 \rangle = 2 \langle \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \rangle$ 。将其添加到 (23) 式中并执行 Grassmann 积分 $\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2$ ，其贡献只包含：

$$2 \times \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \quad (45)$$

因而相应的杂志点张量为：

$$\begin{aligned} & T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)}^{(\bar{\psi} i \gamma_5 \psi)^2} = (-1)^{i_1 \times (i'_2 + i'_1 + j_2 + j_1) + i_2 \times (i'_2 + i'_1 + j_2) + j'_1 \times (i'_2 + i'_1) + j'_2 \times i'_2 + i'_1 + i'_2} \times e^{\frac{\mu}{2}(i_2 + i'_2 - j_2 - j'_2)} \\ & \times \left(\frac{\sqrt{2}}{2} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \times 2 \delta_{i_1+i_2+j'_1+j'_2,0} \delta_{j_1+j_2+i'_1+i'_2,0} \end{aligned}$$

考虑 $\langle \bar{\psi}\psi \rangle = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$ ，这个情况和计算 pseudoscalar condensate 差不多。Grassmann 积分后存在的贡献包括：

$$\bar{\psi}_1\psi_1 \rightarrow -(m+2r)\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \quad (46)$$

$$\bar{\psi}_1\psi_1 \rightarrow (+i)^{j'_2+i_2+i'_2+j_2}(-1)^{i_2+i_1+i'_1+j_2}\delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} \quad (47)$$

$$\bar{\psi}_2\psi_2 \rightarrow -(m+2r)\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \quad (48)$$

$$\bar{\psi}_2\psi_2 \rightarrow \delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} \quad (49)$$

$$\begin{aligned} & T_{(i_1j_1)(i_2j_2)(i'_1j'_1)(i'_2j'_2)}^{\bar{\psi}\psi} = (-1)^{i_1 \times (i'_2+i'_1+j_2+j_1) + i_2 \times (i'_2+i'_1+j_2) + j'_1 \times (i'_2+i'_1) + j'_2 \times i'_2 + i'_1 + i'_2} \times e^{\frac{\mu}{2}(i_2+i'_2-j_2-j'_2)} \\ & \times \left(\frac{\sqrt{2}}{2} \right)^{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2} \times \left[(+i)^{j'_2+i_2+i'_2+j_2}(-1)^{i_2+i_1+i'_1+j_2}\delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} \right. \\ & + \left. \delta_{i_1+i_2+j'_1+j'_2,1}\delta_{j_1+j_2+i'_1+i'_2,1} - 2 \times (m+2r)\delta_{i_1+i_2+j'_1+j'_2,0}\delta_{j_1+j_2+i'_1+i'_2,0} \right] \end{aligned}$$

如果这个作用量 S 包含 lattice anisotropy term:

$$\begin{aligned}
S = & -\frac{1}{2} \sum_{n \in \Lambda_2} \sum_{\nu=1,2} [\gamma^{\delta_{\nu,2}} e^{\mu \delta_{\nu,2}} \bar{\psi}(n) (rI - \gamma_{\nu}) \psi(n + \hat{\nu}) + \gamma^{\delta_{\nu,2}} e^{-\mu \delta_{\nu,2}} \bar{\psi}(n + \hat{\nu}) (rI + \gamma_{\nu}) \psi(n)] \\
+ & (m + 2r) \sum_n \bar{\psi}(n) \psi(n) - \frac{g^2}{2} \sum_n [(\bar{\psi}(n) \psi(n))^2 + (\bar{\psi}(n) i \gamma_5 \psi(n))^2] \quad (50)
\end{aligned}$$

那相应的 pseudoscalar 期望值就会变成:

$$\begin{aligned}
& T_{(i_1 j_1)(i_2 j_2)(i'_1 j'_1)(i'_2 j'_2)}^{\bar{\psi} i \gamma_5 \psi} = (-1)^{i_1 \times (i'_2 + i'_1 + j_2 + j_1) + i_2 \times (i'_2 + i'_1 + j_2) + j'_1 \times (i'_2 + i'_1) + j'_2 \times i'_2 + i'_1 + i'_2} \times e^{\frac{\mu}{2}(i_2 + i'_2 - j_2 - j'_2)} \\
\times & \sqrt{\gamma^{i_2 + i'_2 + j_2 + j'_2}} \times \left(\frac{\sqrt{2}}{2} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \times \left[i(+i)^{j'_2 + i_2 + i'_2 + j_2} (-1)^{i_2 + i_1 + i'_1 + j_2} \delta_{i_1 + i_2 + j'_1 + j'_2, 1} \delta_{j_1 + j_2 + i'_1 + i'_2, 1} \right. \\
- & \left. i \delta_{i_1 + i_2 + j'_1 + j'_2, 1} \delta_{j_1 + j_2 + i'_1 + i'_2, 1} \right]
\end{aligned}$$