

N -flavor Gross-Neveu-Wilson model

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1 Model

N -flavor Gross-Neveu-Wilson model is defined by the following action,

$$\begin{aligned} S &= -\frac{1}{2} \sum_{n \in \Lambda_2} \sum_{\nu=1,2} \left[e^{\mu\delta_{\nu,2}} \bar{\psi}(n)(r\mathbb{1} - \gamma_\nu)\psi(n + \hat{\nu}) + e^{-\mu\delta_{\nu,2}} \bar{\psi}(n + \hat{\nu})(r\mathbb{1} + \gamma_\nu)\psi(n) \right] \\ &\quad + (M + 2r) \sum_n \bar{\psi}(n)\psi(n) - \frac{g_\sigma^2}{2N} \sum_n (\bar{\psi}(n)\psi(n))^2 - \frac{g_\pi^2}{2N} \sum_n (\bar{\psi}(n)i\gamma_5\psi(n))^2 \\ &= -\frac{1}{2} \sum_{f=1}^N \sum_{n \in \Lambda_2} \sum_{\nu=1,2} \left[e^{\mu\delta_{\nu,2}} \bar{\psi}^{(f)}(n)(r\mathbb{1} - \gamma_\nu)\psi^{(f)}(n + \hat{\nu}) + e^{-\mu\delta_{\nu,2}} \bar{\psi}^{(f)}(n + \hat{\nu})(r\mathbb{1} + \gamma_\nu)\psi^{(f)}(n) \right] \\ &\quad + \sum_{f=1}^N \sum_n (M_f + 2r^{(f)}) \bar{\psi}^{(f)}(n)\psi^{(f)}(n) \\ &\quad - \frac{g^2}{2N} \sum_n \left[\left(\sum_f \bar{\psi}^{(f)}(n)\psi^{(f)}(n) \right)^2 - \left(\sum_f \bar{\psi}^{(f)}(n)i\gamma_5\psi^{(f)}(n) \right)^2 \right]. \end{aligned} \tag{1.1}$$

2 $N = 1$

2.1 Pure tensor

$$\begin{aligned} S &= -\frac{1}{2} \sum_{n \in \Lambda_2} \sum_{\nu=1,2} \left[e^{\mu\delta_{\nu,2}} \bar{\psi}(n)(r\mathbb{1} - \gamma_\nu)\psi(n + \hat{\nu}) + e^{-\mu\delta_{\nu,2}} \bar{\psi}(n + \hat{\nu})(r\mathbb{1} + \gamma_\nu)\psi(n) \right] \\ &\quad + (M + 2r) \sum_n \bar{\psi}(n)\psi(n) - \frac{g_\sigma^2}{2N} \sum_n (\bar{\psi}(n)\psi(n))^2 - \frac{g_\pi^2}{2N} \sum_n (\bar{\psi}(n)i\gamma_5\psi(n))^2. \end{aligned} \tag{2.1}$$

Choosing the Dirac-Pauli representation, say $\gamma_1 = \sigma_x$, $\gamma_2 = \sigma_z$, $\gamma_5 = \sigma_y$, and $r = 1$, we have

$$S = - \sum_n [\bar{\phi}_2(n)\phi_2(n+\hat{1}) + \bar{\phi}_1(n+\hat{1})\phi_1(n) + e^\mu \bar{\psi}_2(n)\psi_2(n+\hat{2}) + e^{-\mu} \bar{\psi}_1(n+\hat{2})\psi_1(n)] \\ + \sum_n \left[(M+2)\bar{\psi}_1(n)\psi_1(n) + (M+2)\bar{\psi}_2(n)\psi_2(n) - \frac{g_\sigma^2 + g_\pi^2}{N} \bar{\psi}_1(n)\psi_1(n)\bar{\psi}_2(n)\psi_2(n) \right], \quad (2.2)$$

where we set

$$\phi(n) = H\psi(n), \quad \bar{\phi}(n) = \bar{\psi}(n)H, \quad (2.3)$$

with the Hadamard gate H . On the other hand, choosing the chiral representation, say $\gamma_1 = \sigma_x$, $\gamma_2 = \sigma_y$, $\gamma_5 = -i\gamma_1\gamma_2 = \sigma_z$, we have

$$S = - \sum_n [\bar{\chi}_2(n)\chi_2(n+\hat{1}) + \bar{\chi}_1(n+\hat{1})\chi_1(n) + e^\mu \bar{\chi}_2(n)\chi_2(n+\hat{2}) + e^{-\mu} \bar{\chi}_1(n+\hat{2})\chi_1(n)] \\ + \sum_n \left[(M+2)\bar{\psi}_1(n)\psi_1(n) + (M+2)\bar{\psi}_2(n)\psi_2(n) - \frac{g_\sigma^2 + g_\pi^2}{N} \bar{\psi}_1(n)\psi_1(n)\bar{\psi}_2(n)\psi_2(n) \right], \quad (2.4)$$

where we set

$$\chi(n) = U^\dagger \psi(n), \quad \bar{\chi}(n) = \bar{\psi}(n)U, \quad (2.5)$$

with

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad (2.6)$$

2.1.1 Dirac-Pauli representation

Decompose hopping factors via

$$e^{\bar{\phi}_2(n)\phi_2(n+\hat{1})} = \int \int d\bar{\eta}_1(n)d\eta_1(n) e^{-\bar{\eta}_1(n)\eta_1(n)} e^{\bar{\phi}_2(n)\eta_1(n)} e^{-\phi_2(n+\hat{1})\bar{\eta}_1(n)}, \quad (2.7)$$

$$e^{\bar{\phi}_1(n+\hat{1})\phi_1(n)} = \int \int d\bar{\zeta}_1(n)d\zeta_1(n) e^{-\bar{\zeta}_1(n)\zeta_1(n)} e^{\bar{\phi}_1(n+\hat{1})\bar{\zeta}_1(n)} e^{\phi_1(n)\zeta_1(n)}, \quad (2.8)$$

$$e^{e^\mu \bar{\psi}_2(n)\psi_2(n+\hat{2})} = \int \int d\bar{\eta}_2(n)d\eta_2(n) e^{-\bar{\eta}_2(n)\eta_2(n)} e^{e^{\mu/2} \bar{\psi}_2(n)\eta_2(n)} e^{-e^{\mu/2} \psi_2(n+\hat{2})\bar{\eta}_2(n)}, \quad (2.9)$$

$$e^{e^{-\mu} \bar{\psi}_1(n+\hat{2})\psi_1(n)} = \int \int d\bar{\zeta}_2(n)d\zeta_2(n) e^{-\bar{\zeta}_2(n)\zeta_2(n)} e^{e^{-\mu/2} \bar{\psi}_1(n+\hat{2})\bar{\zeta}_2(n)} e^{e^{-\mu/2} \psi_1(n)\zeta_2(n)}, \quad (2.10)$$

one obtains

$$\begin{aligned} \mathcal{T} = & \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 e^{-(M+2)\bar{\psi}_1\psi_1 - (M+2)\bar{\psi}_2\psi_2 + \frac{g_\sigma^2 + g_\pi^2}{N}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2} \\ & \times e^{\bar{\phi}_2\eta_1(n)} e^{-\phi_2\bar{\eta}_1(n-\hat{1})} e^{\bar{\phi}_1\bar{\zeta}_1(n-\hat{1})} e^{\phi_1\zeta_1(n)} e^{\mu/2\bar{\psi}_2\eta_2(n)} e^{-\mu/2\psi_2\bar{\eta}_2(n-\hat{2})} e^{-\mu/2\bar{\psi}_1\bar{\zeta}_2(n-\hat{2})} e^{-\mu/2\psi_1\zeta_2(n)}. \end{aligned} \quad (2.11)$$

Then

$$\mathcal{T} = \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \eta_1^{i_1} \zeta_1^{j_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\eta}_1^{i'_1} \bar{\zeta}_1^{j'_1} \bar{\eta}_2^{i'_2} \bar{\zeta}_2^{j'_2}, \quad (2.12)$$

with

$$\begin{aligned} & t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \\ & = \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 e^{-(M+2)\bar{\psi}_1\psi_1 - (M+2)\bar{\psi}_2\psi_2 + \frac{g_\sigma^2 + g_\pi^2}{N}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2} \psi_2^{i'_2} \bar{\psi}_1^{j'_2} \phi_2^{i'_1} \bar{\phi}_1^{j'_1} \psi_1^{j_2} \bar{\psi}_2^{i_2} \phi_1^{j_1} \bar{\phi}_2^{i_1}. \end{aligned} \quad (2.13)$$

To carry out the integration is t , it is useful to note that

$$\begin{aligned} & \psi_2^{i'_2} \bar{\psi}_1^{j'_2} \phi_2^{i'_1} \bar{\phi}_1^{j'_1} \psi_1^{j_2} \bar{\psi}_2^{i_2} \phi_1^{j_1} \bar{\phi}_2^{i_1} \\ & = \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \psi_2^{i'_2} \bar{\psi}_1^{j'_2} (\psi_1 - \psi_2)^{i'_1} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} \psi_1^{j_2} \bar{\psi}_2^{i_2} (\psi_1 + \psi_2)^{j_1} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} \\ & = M_{i'_1 j'_1} M_{j_1 i_1} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \\ & \times \psi_2^{i'_2} \bar{\psi}_1^{j'_2} \left[\psi_1^{i'_1} \bar{\psi}_1^{j'_1} + (-\psi_2)^{i'_1} \bar{\psi}_1^{j'_1} + \psi_1^{i'_1} \bar{\psi}_2^{j'_1} + (-\psi_2)^{i'_1} \bar{\psi}_2^{j'_1} \right] \\ & \times \psi_1^{j_2} \bar{\psi}_2^{i_2} \left[\psi_1^{j_1} \bar{\psi}_1^{i_1} + \psi_2^{j_1} \bar{\psi}_1^{i_1} + \psi_1^{j_1} (-\bar{\psi}_2)^{i_1} + \psi_2^{j_1} (-\bar{\psi}_2)^{i_1} \right] \end{aligned} \quad (2.14)$$

$$\begin{aligned} & \psi_2^{i'_2} \bar{\psi}_1^{j'_2} \left[\psi_1^{i'_1} \bar{\psi}_1^{j'_1} + (-\psi_2)^{i'_1} \bar{\psi}_1^{j'_1} + \psi_1^{i'_1} \bar{\psi}_2^{j'_1} + (-\psi_2)^{i'_1} \bar{\psi}_2^{j'_1} \right] \psi_1^{j_2} \bar{\psi}_2^{i_2} \left[\psi_1^{j_1} \bar{\psi}_1^{i_1} + \psi_2^{j_1} \bar{\psi}_1^{i_1} + \psi_1^{j_1} (-\bar{\psi}_2)^{i_1} + \psi_2^{j_1} (-\bar{\psi}_2)^{i_1} \right] \\ & = (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1)}} \bar{\psi}_2^{i_2} \psi_2^{i'_2} \bar{\psi}_1^{j_2 + i_1 + j'_1} \psi_1^{j_2 + j_1 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,1)}} \bar{\psi}_2^{i_2} \psi_2^{i'_2 + i'_1} \bar{\psi}_1^{j_2 + i_1 + j'_1} \psi_1^{j_2 + j_1} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,1)}} \bar{\psi}_2^{i_2 + j'_1} \psi_2^{i'_2} \bar{\psi}_1^{j_2 + i_1} \psi_1^{j_2 + j_1 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,1)}} \bar{\psi}_2^{i_2 + j'_1} \psi_2^{i'_2 + i'_1} \bar{\psi}_1^{j_2 + i_1} \psi_1^{j_2 + j_1} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,2)}} \bar{\psi}_2^{i_2} \psi_2^{i'_2 + j_1} \bar{\psi}_1^{j_2 + i_1 + j'_1} \psi_1^{j_2 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,2)}} \bar{\psi}_2^{i_2} \psi_2^{i'_2 + j_1 + i'_1} \bar{\psi}_1^{j_2 + i_1 + j'_1} \psi_1^{j_2} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,2)}} \bar{\psi}_2^{i_2 + j'_1} \psi_2^{i'_2 + j_1} \bar{\psi}_1^{j_2 + i_1} \psi_1^{j_2 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,2)}} \bar{\psi}_2^{i_2 + j'_1} \psi_2^{i'_2 + j_1 + i'_1} \bar{\psi}_1^{j_2 + i_1} \psi_1^{j_2} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,3)}} \bar{\psi}_2^{i_2 + i_1} \psi_2^{i'_2} \bar{\psi}_1^{j_2 + j'_1} \psi_1^{j_2 + j_1 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,3)}} \bar{\psi}_2^{i_2 + i_1} \psi_2^{i'_2 + i'_1} \bar{\psi}_1^{j_2 + j'_1} \psi_1^{j_2 + j_1} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,3)}} \bar{\psi}_2^{i_2 + i_1 + j'_1} \psi_2^{i'_2} \bar{\psi}_1^{j_2} \psi_1^{j_2 + j_1 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,3)}} \bar{\psi}_2^{i_2 + i_1 + j'_1} \psi_2^{i'_2 + i'_1} \bar{\psi}_1^{j_2} \psi_1^{j_2 + j_1} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,4)}} \bar{\psi}_2^{i_2 + i_1} \psi_2^{i'_2 + j_1} \bar{\psi}_1^{j_2 + j'_1} \psi_1^{j_2 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,4)}} \bar{\psi}_2^{i_2 + i_1} \psi_2^{i'_2 + j_1 + i'_1} \bar{\psi}_1^{j_2 + j'_1} \psi_1^{j_2} \\ & + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,4)}} \bar{\psi}_2^{i_2 + i_1 + j'_1} \psi_2^{i'_2 + j_1} \bar{\psi}_1^{j_2} \psi_1^{j_2 + i'_1} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,4)}} \bar{\psi}_2^{i_2 + i_1 + j'_1} \psi_2^{i'_2 + j_1 + i'_1} \bar{\psi}_1^{j_2} \psi_1^{j_2}. \end{aligned} \quad (2.15)$$

Now, we can easily see

$$\begin{aligned}
& t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \\
&= M_{i'_1 j'_1} M_{j_1 i_1} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \left[\left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(0,0,0,0)} \right. \\
&\quad \left. - (M+2) \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1,0,0)} - (M+2) \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(0,0,1,1)} + \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1,1,1)} \right], \quad (2.16)
\end{aligned}$$

where

$$\begin{aligned}
& \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(a,b,c,d)} \\
&= (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1)}} \delta_{i_2, a} \delta_{i'_2, b} \delta_{j'_2 + i_1 + j'_1, c} \delta_{j_2 + j_1 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,1)}} \delta_{i_2, a} \delta_{i'_2 + i'_1, b} \delta_{j'_2 + i_1 + j'_1, c} \delta_{j_2 + j_1, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,1)}} \delta_{i_2 + j'_1, a} \delta_{i'_2, b} \delta_{j'_2 + i_1, c} \delta_{j_2 + j_1 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,1)}} \delta_{i_2 + j'_1, a} \delta_{i'_2 + i'_1, b} \delta_{j'_2 + i_1, c} \delta_{j_2 + j_1, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,2)}} \delta_{i_2, a} \delta_{i'_2 + j_1, b} \delta_{j'_2 + i_1 + j'_1, c} \delta_{j_2 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,2)}} \delta_{i_2, a} \delta_{i'_2 + j_1 + i'_1, b} \delta_{j'_2 + i_1 + j'_1, c} \delta_{j_2, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,2)}} \delta_{i_2 + j'_1, a} \delta_{i'_2 + j_1, b} \delta_{j'_2 + i_1, c} \delta_{j_2 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,2)}} \delta_{i_2 + j'_1, a} \delta_{i'_2 + j_1 + i'_1, b} \delta_{j'_2 + i_1, c} \delta_{j_2, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,3)}} \delta_{i_2 + i_1, a} \delta_{i'_2, b} \delta_{j'_2 + j'_1, c} \delta_{j_2 + j_1 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,3)}} \delta_{i_2 + i_1, a} \delta_{i'_2 + i'_1, b} \delta_{j'_2 + j'_1, c} \delta_{j_2 + j_1, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,3)}} \delta_{i_2 + i_1 + j'_1, a} \delta_{i'_2, b} \delta_{j'_2, c} \delta_{j_2 + j_1 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,3)}} \delta_{i_2 + i_1 + j'_1, a} \delta_{i'_2 + i'_1, b} \delta_{j'_2, c} \delta_{j_2 + j_1, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,4)}} \delta_{i_2 + i_1, a} \delta_{i'_2 + j_1, b} \delta_{j'_2 + j'_1, c} \delta_{j_2 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,4)}} \delta_{i_2 + i_1, a} \delta_{i'_2 + j_1 + i'_1, b} \delta_{j'_2 + j'_1, c} \delta_{j_2, d} \\
&+ (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,4)}} \delta_{i_2 + i_1 + j'_1, a} \delta_{i'_2 + j_1, b} \delta_{j'_2, c} \delta_{j_2 + i'_1, d} + (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,4)}} \delta_{i_2 + i_1 + j'_1, a} \delta_{i'_2 + j_1 + i'_1, b} \delta_{j'_2, c} \delta_{j_2, d}. \quad (2.17)
\end{aligned}$$

Setting

$$T_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} = (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}, \quad (2.18)$$

we finally obtain the Grassmann tensor as

$$\mathcal{T} = \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} T_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \eta_1^{i_1} \zeta_1^{j_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\eta}_1^{i'_1} \bar{\zeta}_1^{j'_1} \bar{\eta}_2^{i'_2} \bar{\zeta}_2^{j'_2}. \quad (2.19)$$

Notice that reordering factor $P^{(a,b)}$'s are defined as follows,

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1)} = i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + j'_1 i'_1 + i_1(i'_1 + j_2 + j_1),$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,1)} = i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + i'_1 j'_2 + i_1(j_2 + j_1) + i'_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,1)} = j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i_1(i'_1 + j_2 + j_1),$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,1)} = j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i'_1 j'_2 + i_1(j_2 + j_1) + i'_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,2)} = i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + j_1(j'_2 + i'_1 + j'_1 + j_2) + j'_1 i'_1 + i_1(i'_1 + j_2),$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,2)} = i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + i'_1 j'_2 + j_1(j'_2 + j'_1 + j_2) + i_1 j_2 + i'_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,2)} = j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + j_1(j'_2 + i'_1 + j_2) + i_1(i'_1 + j_2),$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,2)} = j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i'_1 j'_2 + j_1(j'_2 + j_2) + i_1 j_2 + i'_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,3)} = i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j'_1 + j_2 + j_1) + j'_1 i'_1 + i_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,3)} = i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j'_1 + j_2 + j_1) + i'_1 j'_2 + i'_1 + i_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,3)} = j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j_2 + j_1) + i_1,$$

$$P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,3)} = j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j_2 + j_1) + i'_1 j'_2 + i'_1 + i_1,$$

$$\begin{aligned} P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,4)} &= i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j'_1 + j_2 + j_1) \\ &\quad + j_1(j'_2 + i'_1 + j'_1 + j_2) + j'_1 i'_1 + i_1, \end{aligned}$$

$$\begin{aligned} P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,4)} &= i_2(i'_2 + j'_2 + i'_1 + j'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j'_1 + j_2 + j_1) \\ &\quad + i'_1 j'_2 + j_1(j'_2 + j'_1 + j_2) + i'_1 + i_1, \end{aligned}$$

$$\begin{aligned} P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(3,4)} &= j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j_2 + j_1) \\ &\quad + j_1(j'_2 + i'_1 + j_2) + i_1, \end{aligned}$$

$$\begin{aligned} P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(4,4)} &= j'_1(i'_2 + j'_2 + i'_1) + i_2(i'_2 + j'_2 + i'_1 + j_2) + i_1(i'_2 + j'_2 + i'_1 + j_2 + j_1) \\ &\quad + i'_1 j'_2 + j_1(j'_2 + j_2) + i'_1 + i_1. \end{aligned}$$

2.1.2 Remark

Let us consider the effect of the Hadamard gate applied for the $\hat{1}$ -directional hopping terms. Without the Hadamard gate, one naively decompose these hopping terms via

$$e^{\bar{\psi}_1(n)\psi_1(n+\hat{1})/2} = \int \int d\bar{\alpha}_1(n)d\alpha_1(n) e^{-\bar{\alpha}_1(n)\alpha_1(n)} e^{\bar{\psi}_1(n)\alpha_1(n)/\sqrt{2}} e^{-\psi_1(n+\hat{1})\bar{\alpha}_1(n)/\sqrt{2}}, \quad (2.20)$$

$$e^{-\bar{\psi}_2(n)\psi_1(n+\hat{1})/2} = \int \int d\bar{\beta}_1(n)d\beta_1(n) e^{-\bar{\beta}_1(n)\beta_1(n)} e^{\bar{\psi}_2(n)\beta_1(n)/\sqrt{2}} e^{\psi_1(n+\hat{1})\bar{\beta}_1(n)/\sqrt{2}}, \quad (2.21)$$

$$e^{-\bar{\psi}_1(n)\psi_2(n+\hat{1})/2} = \int \int d\bar{\rho}_1(n)d\rho_1(n) e^{-\bar{\rho}_1(n)\rho_1(n)} e^{\bar{\psi}_1(n)\rho_1(n)/\sqrt{2}} e^{\psi_2(n+\hat{1})\bar{\rho}_1(n)/\sqrt{2}}, \quad (2.22)$$

$$e^{\bar{\psi}_2(n)\psi_2(n+\hat{1})/2} = \int \int d\bar{\sigma}_1(n)d\sigma_1(n) e^{-\bar{\sigma}_1(n)\sigma_1(n)} e^{\bar{\psi}_2(n)\sigma_1(n)/\sqrt{2}} e^{-\psi_2(n+\hat{1})\bar{\sigma}_1(n)/\sqrt{2}}, \quad (2.23)$$

$$e^{\bar{\psi}_1(n+\hat{1})\psi_1(n)/2} = \int \int d\bar{\eta}_1(n)d\eta_1(n) e^{-\bar{\eta}_1(n)\eta_1(n)} e^{\bar{\psi}_1(n+\hat{1})\eta_1(n)/\sqrt{2}} e^{\psi_1(n)\bar{\eta}_1(n)/\sqrt{2}}, \quad (2.24)$$

$$e^{\bar{\psi}_2(n+\hat{1})\psi_1(n)/2} = \int \int d\bar{\zeta}_1(n)d\zeta_1(n) e^{-\bar{\zeta}_1(n)\zeta_1(n)} e^{\bar{\psi}_2(n+\hat{1})\zeta_1(n)/\sqrt{2}} e^{\psi_1(n)\bar{\zeta}_1(n)/\sqrt{2}}, \quad (2.25)$$

$$e^{\bar{\psi}_1(n+\hat{1})\psi_2(n)/2} = \int \int d\bar{\lambda}_1(n)d\lambda_1(n) e^{-\bar{\lambda}_1(n)\lambda_1(n)} e^{\bar{\psi}_1(n+\hat{1})\lambda_1(n)/\sqrt{2}} e^{\psi_2(n)\bar{\lambda}_1(n)/\sqrt{2}}, \quad (2.26)$$

$$e^{\bar{\psi}_2(n+\hat{1})\psi_2(n)/2} = \int \int d\bar{\omega}_1(n)d\omega_1(n) e^{-\bar{\omega}_1(n)\omega_1(n)} e^{\bar{\psi}_2(n+\hat{1})\omega_1(n)/\sqrt{2}} e^{\psi_2(n)\bar{\omega}_1(n)/\sqrt{2}}, \quad (2.27)$$

instead of Eqs. (2.8) and (2.7). As a result, we have

$$\begin{aligned} \mathcal{T} = & \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 e^{-(M+2)\bar{\psi}_1\psi_1 - (M+2)\bar{\psi}_2\psi_2 + \frac{g_\sigma^2 + g_\pi^2}{N}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2} \\ & \times e^{\bar{\psi}_1\alpha_1(n)/\sqrt{2}} e^{-\psi_1\bar{\alpha}_1(n-\hat{1})/\sqrt{2}} e^{\bar{\psi}_2\beta_1(n)/\sqrt{2}} e^{\psi_1\bar{\beta}_1(n-\hat{1})/\sqrt{2}} e^{\bar{\psi}_1\rho_1(n)/\sqrt{2}} e^{\psi_2\bar{\rho}_1(n-\hat{1})/\sqrt{2}} e^{\bar{\psi}_2\sigma_1(n)/\sqrt{2}} e^{-\psi_2\bar{\sigma}_1(n-\hat{1})/\sqrt{2}} \\ & \times e^{\bar{\psi}_1\bar{\eta}_1(n-\hat{1})/\sqrt{2}} e^{\psi_1\eta_1(n)/\sqrt{2}} e^{\bar{\psi}_2\bar{\zeta}_1(n-\hat{1})/\sqrt{2}} e^{\psi_1\zeta_1(n)/\sqrt{2}} e^{\bar{\psi}_1\bar{\lambda}_1(n-\hat{1})/\sqrt{2}} e^{\psi_2\lambda_1(n)/\sqrt{2}} e^{\bar{\psi}_2\bar{\omega}_1(n-\hat{1})/\sqrt{2}} e^{\psi_2\omega_1(n)/\sqrt{2}} \\ & \times e^{e^{\mu/2}\bar{\psi}_2\eta_2(n)} e^{-e^{\mu/2}\psi_2\bar{\eta}_2(n-\hat{2})} e^{-e^{\mu/2}\bar{\psi}_1\bar{\zeta}_2(n-\hat{2})} e^{e^{-\mu/2}\psi_1\zeta_2(n)}. \end{aligned} \quad (2.28)$$

Then

$$\begin{aligned} \mathcal{T} = & \sum_{i_1, j_1, k_1, l_1, m_1, n_1, o_1, p_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1, k'_1, l'_1, m'_1, n'_1, o'_1, p'_1} \sum_{i'_2, j'_2} \\ & \times (-1)^{i'_1 + l'_1 + i'_2} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + k_1 + l_1 + m_1 + n_1 + o_1 + p_1} \left(\frac{1}{\sqrt{2}} \right)^{i'_1 + j'_1 + k'_1 + l'_1 + m'_1 + n'_1 + o'_1 + p'_1} \\ & \times \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] t_{i_1 j_1 k_1 l_1 m_1 n_1 o_1 p_1 i_2 j_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 o'_1 p'_1 i'_2 j'_2} \\ & \times \alpha_1^{i_1} \beta_1^{j_1} \rho_1^{k_1} \sigma_1^{l_1} \eta_1^{m_1} \zeta_1^{n_1} \lambda_1^{o_1} \omega_1^{p_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\omega}_1^{p'_1} \bar{\lambda}_1^{o'_1} \bar{\zeta}_1^{n'_1} \bar{\eta}_1^{m'_1} \bar{\sigma}_1^{l'_1} \bar{\rho}_1^{k'_1} \bar{\beta}_1^{j'_1} \bar{\alpha}_1^{i'_1} \bar{\zeta}_2^{j'_2} \bar{\eta}_2^{i'_2}, \end{aligned} \quad (2.29)$$

with

$$\begin{aligned}
& t_{i_1 j_1 k_1 l_1 m_1 n_1 o_1 p_1 i_2 j_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 o'_1 p'_1 i'_2 j'_2} \\
&= \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 e^{-(M+2)\bar{\psi}_1 \psi_1 - (M+2)\bar{\psi}_2 \psi_2 + \frac{g_\sigma^2 + g_\pi^2}{N} \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2} \\
&\times \psi_2^{i'_2} \bar{\psi}_1^{j'_2} \psi_1^{i'_1} \bar{\psi}_1^{j'_1} \psi_2^{k'_1} \bar{\psi}_1^{l'_1} \bar{\psi}_1^{m'_1} \bar{\psi}_2^{n'_1} \bar{\psi}_1^{o'_1} \bar{\psi}_2^{p'_1} \bar{\psi}_1^{j_2} \bar{\psi}_2^{i_2} \psi_2^{p_1} \psi_2^{o_1} \psi_1^{n_1} \psi_1^{m_1} \bar{\psi}_2^{l_1} \bar{\psi}_1^{k_1} \bar{\psi}_2^{j_1} \bar{\psi}_1^{i_1} \\
&= (-1)^{P_{i_1 j_1 k_1 l_1 m_1 n_1 o_1 p_1 i_2 j_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 o'_1 p'_1 i'_2 j'_2}} \\
&\times \left[\left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \delta_{n'_1 + p'_1 + i_2 + l_1 + j_1, 0} \delta_{i'_2 + k'_1 + l'_1 + p_1 + o_1, 0} \delta_{j'_2 + m'_1 + o'_1 + k_1 + i_1, 0} \delta_{i'_1 + j'_1 + j_2 + n_1 + m_1, 0} \right. \\
&- (M+2) \delta_{n'_1 + p'_1 + i_2 + l_1 + j_1, 1} \delta_{i'_2 + k'_1 + l'_1 + p_1 + o_1, 1} \delta_{j'_2 + m'_1 + o'_1 + k_1 + i_1, 0} \delta_{i'_1 + j'_1 + j_2 + n_1 + m_1, 0} \\
&- (M+2) \delta_{n'_1 + p'_1 + i_2 + l_1 + j_1, 0} \delta_{i'_2 + k'_1 + l'_1 + p_1 + o_1, 0} \delta_{j'_2 + m'_1 + o'_1 + k_1 + i_1, 1} \delta_{i'_1 + j'_1 + j_2 + n_1 + m_1, 1} \\
&\left. + \delta_{n'_1 + p'_1 + i_2 + l_1 + j_1, 1} \delta_{i'_2 + k'_1 + l'_1 + p_1 + o_1, 1} \delta_{j'_2 + m'_1 + o'_1 + k_1 + i_1, 1} \delta_{i'_1 + j'_1 + j_2 + n_1 + m_1, 1} \right], \tag{2.30}
\end{aligned}$$

with

$$\begin{aligned}
& P_{i_1 j_1 k_1 l_1 m_1 n_1 o_1 p_1 i_2 j_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 o'_1 p'_1 i'_2 j'_2} \\
&= n'_1(i'_2 + j'_2 + i'_1 + j'_1 + k'_1 + l'_1 + m'_1) \\
&+ p'_1(i'_2 + j'_2 + i'_1 + j'_1 + k'_1 + l'_1 + m'_1 + o'_1) \\
&+ i_2(i'_2 + j'_2 + i'_1 + j'_1 + k'_1 + l'_1 + m'_1 + o'_1 + j_2) \\
&+ l_1(i'_2 + j'_2 + i'_1 + j'_1 + k'_1 + l'_1 + m'_1 + o'_1 + j_2 + p_1 + o_1 + n_1 + m_1) \\
&+ j_1(i'_2 + j'_2 + i'_1 + j'_1 + k'_1 + l'_1 + m'_1 + o'_1 + j_2 + p_1 + o_1 + n_1 + m_1 + k_1) \\
&+ (k'_1 + l'_1)(j'_2 + i'_1 + j'_1) \\
&+ (p_1 + o_1)(j'_2 + i'_1 + j'_1 + m'_1 + o'_1 + j_2) \\
&+ (m'_1 + o'_1)(i'_1 + j'_1) \\
&+ (k_1 + i_1)(i'_1 + j'_1 + j_2 + n_1 + m_1). \tag{2.31}
\end{aligned}$$

2.1.3 Chiral representation

Decompose hopping factors via

$$e^{\bar{\phi}_2(n)\phi_2(n+\hat{1})} = \int \int d\bar{\eta}_1(n) d\eta_1(n) e^{-\bar{\eta}_1(n)\eta_1(n)} e^{\bar{\phi}_2(n)\eta_1(n)} e^{-\phi_2(n+\hat{1})\bar{\eta}_1(n)}, \tag{2.32}$$

$$e^{\bar{\phi}_1(n+\hat{1})\phi_1(n)} = \int \int d\bar{\zeta}_1(n) d\zeta_1(n) e^{-\bar{\zeta}_1(n)\zeta_1(n)} e^{\bar{\phi}_1(n+\hat{1})\bar{\zeta}_1(n)} e^{\phi_1(n)\zeta_1(n)}, \tag{2.33}$$

$$e^{e^\mu \bar{\chi}_2(n)\chi_2(n+\hat{2})} = \int \int d\bar{\eta}_2(n) d\eta_2(n) e^{-\bar{\eta}_2(n)\eta_2(n)} e^{e^{\mu/2} \bar{\chi}_2(n)\eta_2(n)} e^{-e^{\mu/2} \chi_2(n+\hat{2})\bar{\eta}_2(n)}, \tag{2.34}$$

$$e^{e^{-\mu} \bar{\chi}_1(n+\hat{2})\chi_1(n)} = \int \int d\bar{\zeta}_2(n) d\zeta_2(n) e^{-\bar{\zeta}_2(n)\zeta_2(n)} e^{e^{-\mu/2} \bar{\chi}_1(n+\hat{2})\bar{\zeta}_2(n)} e^{e^{-\mu/2} \chi_1(n)\zeta_2(n)}, \tag{2.35}$$

one obtains

$$\begin{aligned} \mathcal{T} = & \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 e^{-(M+2)\bar{\psi}_1\psi_1 - (M+2)\bar{\psi}_2\psi_2 + \frac{g_\sigma^2 + g_\pi^2}{N}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2} \\ & \times e^{\bar{\phi}_2\eta_1(n)} e^{-\phi_2\bar{\eta}_1(n-\hat{1})} e^{\bar{\phi}_1\bar{\zeta}_1(n-\hat{1})} e^{\phi_1\zeta_1(n)} e^{\mu/2\bar{\chi}_2\eta_2(n)} e^{-\mu/2\chi_2\bar{\eta}_2(n-\hat{2})} e^{\mu/2\bar{\chi}_1\bar{\zeta}_2(n-\hat{2})} e^{-\mu/2\chi_1\zeta_2(n)}. \end{aligned} \quad (2.36)$$

Then

$$\begin{aligned} \mathcal{T} = & \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \zeta_1^{j_1} \zeta_2^{j_2} \bar{\eta}_1^{i'_1} \bar{\eta}_2^{i'_2} \eta_1^{i_1} \eta_2^{i_2} \bar{\zeta}_1^{j'_1} \bar{\zeta}_2^{j'_2} \\ = & \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}} (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \\ & \times \eta_1^{i_1} \zeta_1^{j_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\zeta}_1^{j'_1} \bar{\eta}_1^{i'_1} \bar{\zeta}_2^{j'_2} \bar{\eta}_2^{i'_2}, \end{aligned} \quad (2.37)$$

with

$$\begin{aligned} P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} = & i_1(j_1 + j_2 + i'_1 + i'_2) \\ & + i_2(j_2 + i'_1 + i'_2) \\ & + j'_1(i'_1 + i'_2) \\ & + j'_2 i'_2 \end{aligned} \quad (2.38)$$

and

$$\begin{aligned} & t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \\ = & \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \\ & \times e^{-(M+2)\bar{\psi}_1\psi_1 - (M+2)\bar{\psi}_2\psi_2 + \frac{g_\sigma^2 + g_\pi^2}{N}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2} \bar{\chi}_1^{j_2} \bar{\phi}_1^{j'_1} \bar{\chi}_2^{i_2} \bar{\phi}_2^{i'_1} \chi_2^{i'_2} \phi_2^{i'_1} \chi_1^{j_2} \phi_1^{j_1}. \end{aligned} \quad (2.39)$$

Note that

$$\begin{aligned} \bar{\chi}_1^{j_2} \bar{\phi}_1^{j'_1} \bar{\chi}_2^{i_2} \bar{\phi}_2^{i'_1} \chi_2^{i'_2} \phi_2^{i'_1} \chi_1^{j_2} \phi_1^{j_1} = & \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \\ & \times (\bar{\psi}_1 + i\bar{\psi}_2)^{j_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i'_1} (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1} \end{aligned} \quad (2.40)$$

and all we have to do is to evaluate the Grassmann integral,

$$\begin{aligned} & t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} \\ = & \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \left[1 - (M+2)\bar{\psi}_1\psi_1 - (M+2)\bar{\psi}_2\psi_2 + \left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 \right] \\ & \times \bar{\chi}_1^{j_2} \bar{\phi}_1^{j'_1} \bar{\chi}_2^{i_2} \bar{\phi}_2^{i'_1} \chi_2^{i'_2} \phi_2^{i'_1} \chi_1^{j_2} \phi_1^{j_1}. \end{aligned} \quad (2.41)$$

Let us separately consider Eq. (2.41) by each term. Firstly,

$$\begin{aligned}
& \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \bar{\chi}_1^{j'_2} \bar{\phi}_1^{j'_1} \bar{\chi}_2^{i_2} \bar{\phi}_2^{i_1} \chi_2^{i'_2} \phi_2^{i'_1} \chi_1^{j_2} \phi_1^{j_1} \\
&= - \int \int d\bar{\psi}_2 d\bar{\psi}_1 (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} \\
&\times \int \int d\psi_2 d\psi_1 (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1}
\end{aligned} \tag{2.42}$$

It may be useful to consider

$$\bar{A}_{j'_2 j'_1 i_2 i_1} = \int \int d\bar{\psi}_2 d\bar{\psi}_1 (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1}, \tag{2.43}$$

$$A_{i'_2 i'_1 j_2 j_1} = \int \int d\psi_2 d\psi_1 (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1}. \tag{2.44}$$

Secondly,

$$\begin{aligned}
& - (M+2) \int \int d\psi_2 d\bar{\psi}_2 \int \int d\psi_1 d\bar{\psi}_1 \bar{\psi}_1 \psi_1 \\
& \times (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1} \\
&= - (M+2) (-1)^{i_2+i_1+i'_1+j_2} (+i)^{j'_2+i_2+i'_2+j_2} \int \int d\psi_2 d\bar{\psi}_2 (\bar{\psi}_2)^{j'_2+j'_1+i_2+i_1} (\psi_2)^{i'_2+i'_1+j_2+j_1} \\
&= - (M+2) (-1)^{i_2+i_1+i'_1+j_2} (+i)^{j'_2+i_2+i'_2+j_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1}.
\end{aligned} \tag{2.45}$$

Similarly,

$$\begin{aligned}
& - (M+2) \int \int d\psi_2 d\bar{\psi}_2 \bar{\psi}_2 \psi_2 \int \int d\psi_1 d\bar{\psi}_1 \\
& \times (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1} \\
&= - (M+2) \int \int d\psi_1 d\bar{\psi}_1 (\bar{\psi}_1)^{j'_2+j'_1+i_2+i_1} (\psi_1)^{i'_2+i'_1+j_2+j_1} \\
&= - (M+2) \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1}.
\end{aligned} \tag{2.46}$$

Finally,

$$\begin{aligned}
& \left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \\
& \times (\bar{\psi}_1 + i\bar{\psi}_2)^{j'_2} (\bar{\psi}_1 + \bar{\psi}_2)^{j'_1} (\bar{\psi}_1 - i\bar{\psi}_2)^{i_2} (\bar{\psi}_1 - \bar{\psi}_2)^{i_1} (\psi_1 + i\psi_2)^{i'_2} (\psi_1 - \psi_2)^{i'_1} (\psi_1 - i\psi_2)^{j_2} (\psi_1 + \psi_2)^{j_1} \\
&= \left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0}.
\end{aligned} \tag{2.47}$$

Totally, we obtain

$$\begin{aligned}
& t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2} / \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \\
&= -A_{i'_2 i'_1 j_2 j_1} \bar{A}_{j'_2 j'_1 i_2 i_1} \\
&- (M+2)(-1)^{i_2 + i_1 + i'_1 + j_2} (+i)^{j'_2 + i_2 + i'_2 + j_2} \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \\
&- (M+2) \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \\
&+ \left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0}.
\end{aligned} \tag{2.48}$$

2.2 Impurity tensor

2.2.1 Dirac-Pauli representation

Here, we consider the tensor network representations for $\langle \bar{\psi} \psi \rangle$ and $\langle \bar{\psi} i \gamma_5 \psi \rangle$. For $k, l = 1, 2$,

$$\begin{aligned}
\mathcal{S}^{(k,l)} &= \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \bar{\psi}_k \psi_l \left[e^{-(M+2)\bar{\psi}_k \psi_l} \delta_{k,l} + (1 - \delta_{k,l}) \right] \\
&\times e^{\bar{\phi}_2 \eta_1(n)} e^{-\phi_2 \bar{\eta}_1(n-\hat{1})} e^{\bar{\phi}_1 \bar{\zeta}_1(n-\hat{1})} e^{\phi_1 \zeta_1(n)} e^{e^{\mu/2} \bar{\psi}_2 \eta_2(n)} e^{-e^{\mu/2} \psi_2 \bar{\eta}_2(n-\hat{2})} e^{e^{-\mu/2} \bar{\psi}_1 \bar{\zeta}_2(n-\hat{2})} e^{-e^{-\mu/2} \psi_1 \zeta_2(n)}.
\end{aligned} \tag{2.49}$$

$$\mathcal{S}^{(k,l)} = \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k,l)} \eta_1^{i_1} \zeta_1^{j_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\zeta}_1^{j'_1} \bar{\eta}_1^{i'_1} \bar{\zeta}_2^{j'_2} \bar{\eta}_2^{i'_2}, \tag{2.50}$$

where

$$\begin{aligned}
& s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k,l)} \\
&= \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \bar{\psi}_k \psi_l \left[e^{-(M+2)\bar{\psi}_k \psi_l} \delta_{k,l} + (1 - \delta_{k,l}) \right] \psi_2^{i'_2} \bar{\psi}_1^{j'_2} \phi_2^{i'_1} \bar{\phi}_1^{j'_1} \psi_1^{j_2} \bar{\psi}_2^{i_2} \phi_1^{j_1} \bar{\phi}_2^{i_1}.
\end{aligned} \tag{2.51}$$

We have

$$s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1)} = M_{i'_1 j'_1} M_{j_1 i_1} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \left[-(M+2) \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(0,0,0,0)} + \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,1,0,0)} \right], \tag{2.52}$$

$$s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,2)} = M_{i'_1 j'_1} M_{j_1 i_1} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \left[-(M+2) \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(0,0,0,0)} + \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(0,0,1,1)} \right], \tag{2.53}$$

$$s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,2)} = M_{i'_1 j'_1} M_{j_1 i_1} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1,0,0,1)}, \tag{2.54}$$

$$s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2,1)} = M_{i'_1 j'_1} M_{j_1 i_1} \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i'_1 + j'_1} \Delta_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(0,1,1,0)}. \quad (2.55)$$

Setting

$$S_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k,l)} = (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k,l)}, \quad (2.56)$$

we finally obtain the impurity Grassmann tensor as

$$\mathcal{S}^{(k,l)} = \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} S_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k,l)} \eta_1^{i_1} \zeta_1^{j_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\zeta}_1^{j'_1} \bar{\eta}_1^{i'_1} \bar{\zeta}_2^{j'_2} \bar{\eta}_2^{i'_2}. \quad (2.57)$$

2.2.2 Chiral representation

With this representation, both $\langle \bar{\psi} \psi \rangle$ and $\langle \bar{\psi} i \gamma_5 \psi \rangle$ are diagonal in the Dirac space. It is sufficient to consider

$$\begin{aligned} & s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k)} \\ &= \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \bar{\psi}_k \psi_k \\ &\times \left[1 - (M+2) \bar{\psi}_1 \psi_1 - (M+2) \bar{\psi}_2 \psi_2 + \left((M+2)^2 + \frac{g_\sigma^2 + g_\pi^2}{N} \right) \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \right] \\ &\times \bar{\chi}_1^{j'_2} \bar{\phi}_1^{j'_1} \bar{\chi}_2^{i_2} \bar{\phi}_2^{i_1} \chi_2^{i'_2} \phi_2^{i'_1} \chi_1^{j_2} \phi_1^{j_1} \\ &= \int \int \int \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 \bar{\psi}_k \psi_k \left[1 - (M+2) \bar{\psi}_k \psi_k \right] \bar{\chi}_1^{j'_2} \bar{\phi}_1^{j'_1} \bar{\chi}_2^{i_2} \bar{\phi}_2^{i_1} \chi_2^{i'_2} \phi_2^{i'_1} \chi_1^{j_2} \phi_1^{j_1} \end{aligned} \quad (2.58)$$

for $k = 1, 2$. This integration is immediately evaluated as

$$\begin{aligned} & s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(1)} / \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \\ &= (-1)^{i_2 + i_1 + i'_1 + j_2} (+i)^{j'_2 + i_2 + i'_2 + j_2} \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \\ &- (M+2) \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0}, \end{aligned} \quad (2.59)$$

and

$$\begin{aligned} & s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(2)} / \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \\ &= \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \\ &- (M+2) \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0}. \end{aligned} \quad (2.60)$$

Setting

$$S_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k)} = (-1)^{P_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}} (-1)^{i'_1 + i'_2} \exp \left[\frac{\mu}{2} (i_2 - j_2 + i'_2 - j'_2) \right] s_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k)}, \quad (2.61)$$

we finally obtain the impurity Grassmann tensor as

$$\mathcal{S}^{(k)} = \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i'_1, j'_1} \sum_{i'_2, j'_2} S_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2}^{(k)} \eta_1^{i_1} \zeta_1^{j_1} \eta_2^{i_2} \zeta_2^{j_2} \bar{\zeta}_1^{j'_1} \bar{\eta}_1^{i'_1} \bar{\zeta}_2^{j'_2} \bar{\eta}_2^{i'_2}. \quad (2.62)$$

3 $N_f = 2$

Since the hopping term is diagonal in the flavor space, there is no difficulty to derive the tensor network representation for $N \neq 1$. All we have to consider is a kind of combinatorics!

3.1 Pure tensor

$$\begin{aligned}
S = & -\frac{1}{2} \sum_{f=1}^N \sum_{n \in \Lambda_2} \sum_{\nu=1,2} [e^{\mu_f \delta_{\nu,2}} \bar{\psi}^{(f)}(n) (r\mathbb{1} - \gamma_\nu) \psi^{(f)}(n + \hat{\nu}) + e^{-\mu_f \delta_{\nu,2}} \bar{\psi}^{(f)}(n + \hat{\nu}) (r\mathbb{1} + \gamma_\nu) \psi^{(f)}(n)] \\
& + \sum_{f=1} \sum_n (M_f + 2r^{(f)}) \bar{\psi}^{(f)}(n) \psi^{(f)}(n) \\
& - \frac{g_\sigma^2}{2N} \sum_n \left(\sum_f \bar{\psi}^{(f)}(n) \psi^{(f)}(n) \right)^2 - \frac{g_\pi^2}{2N} \sum_n \left(\sum_f \bar{\psi}^{(f)}(n) i\gamma_5 \psi^{(f)}(n) \right)^2 \\
& + h_a \sum_{f,f'} \sum_n \bar{\psi}^{(f)}(n) i\gamma_5 (\tau_a)_{ff'} \psi^{(f')}(n). \tag{3.1}
\end{aligned}$$

We assume the chiral representation for the γ -matrices. For all f , we set $r^{(f)} = 1$. By repeating the decomposition introduced in the case of $N = 1$,

$$e^{\bar{\phi}_2^{(f)}(n) \phi_2^{(f)}(n+\hat{1})} = \int \int d\bar{\eta}_1^{(f)}(n) d\eta_1^{(f)}(n) e^{-\bar{\eta}_1^{(f)}(n) \eta_1^{(f)}(n)} e^{\bar{\phi}_2^{(f)}(n) \eta_1^{(f)}(n)} e^{-\phi_2^{(f)}(n+\hat{1}) \bar{\eta}_1^{(f)}(n)}, \tag{3.2}$$

$$e^{\bar{\phi}_1^{(f)}(n+\hat{1}) \phi_1^{(f)}(n)} = \int \int d\bar{\zeta}_1^{(f)}(n) d\zeta_1^{(f)}(n) e^{-\bar{\zeta}_1^{(f)}(n) \zeta_1^{(f)}(n)} e^{\bar{\phi}_1^{(f)}(n+\hat{1}) \bar{\zeta}_1^{(f)}(n)} e^{\phi_1^{(f)}(n) \zeta_1^{(f)}(n)}, \tag{3.3}$$

$$e^{e^{\mu_f} \bar{\chi}_2^{(f)}(n) \chi_2^{(f)}(n+\hat{2})} = \int \int d\bar{\eta}_2^{(f)}(n) d\eta_2^{(f)}(n) e^{-\bar{\eta}_2^{(f)}(n) \eta_2^{(f)}(n)} e^{e^{\mu_f/2} \bar{\chi}_2^{(f)}(n) \eta_2^{(f)}(n)} e^{-e^{\mu_f/2} \chi_2^{(f)}(n+\hat{2}) \bar{\eta}_2^{(f)}(n)}, \tag{3.4}$$

$$e^{e^{-\mu_f} \bar{\chi}_1^{(f)}(n+\hat{2}) \chi_1^{(f)}(n)} = \int \int d\bar{\zeta}_2^{(f)}(n) d\zeta_2^{(f)}(n) e^{-\bar{\zeta}_2^{(f)}(n) \zeta_2^{(f)}(n)} e^{e^{-\mu_f/2} \bar{\chi}_1^{(f)}(n+\hat{2}) \bar{\zeta}_2^{(f)}(n)} e^{-e^{-\mu_f/2} \chi_1^{(f)}(n) \zeta_2^{(f)}(n)}, \tag{3.5}$$

one obtains

$$\begin{aligned}
\mathcal{T} = & \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} e^{-(M_f+2)\bar{\psi}_1^{(f)}\psi_1^{(f)} - (M_f+2)\bar{\psi}_2^{(f)}\psi_2^{(f)}} \right) \\
& \times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{4} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right)^2 + \frac{g_\sigma^2 - g_\pi^2}{4} \left(\sum_f \bar{\psi}_2^{(f)} \psi_2^{(f)} \right)^2 \right] \\
& \times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right) \left(\sum_{f'} \bar{\psi}_2^{(f')} \psi_2^{(f')} \right) \right] \\
& \times \exp \left[-i h_a \sum_{f, f'} \left(\bar{\psi}_1^{(f)} (\tau_a)^{ff'} \psi_1^{(f')} - \bar{\psi}_2^{(f)} (\tau_a)^{ff'} \psi_2^{(f')} \right) \right] \\
& \times \prod_f e^{\bar{\phi}_2^{(f)} \eta_1^{(f)}(n)} e^{-\phi_2^{(f)} \bar{\eta}_1^{(f)}(n-\hat{1})} e^{\bar{\phi}_1^{(f)} \bar{\zeta}_1^{(f)}(n-\hat{1})} e^{\phi_1^{(f)} \zeta_1^{(f)}(n)} \\
& \times e^{e^{\mu f/2} \bar{\chi}_2^{(f)} \eta_2^{(f)}(n)} e^{-e^{\mu f/2} \chi_2^{(f)} \bar{\eta}_2^{(f)}(n-\hat{2})} e^{-e^{\mu f/2} \bar{\chi}_1^{(f)} \bar{\zeta}_2^{(f)}(n-\hat{2})} e^{-e^{\mu f/2} \chi_1^{(f)} \zeta_2^{(f)}(n)} \\
= & \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} \right) \\
& \times \exp \left[- \sum_{f, f'} \bar{\psi}_1^{(f)} \left\{ (M_f + 2) \delta_{ff'} + i h_a (\tau_a)^{ff'} \right\} \psi_1^{(f')} \right] \\
& \times \exp \left[- \sum_{f, f'} \bar{\psi}_2^{(f)} \left\{ (M_f + 2) \delta_{ff'} - i h_a (\tau_a)^{ff'} \right\} \psi_2^{(f')} \right] \\
& \times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{4} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right)^2 + \frac{g_\sigma^2 - g_\pi^2}{4} \left(\sum_f \bar{\psi}_2^{(f)} \psi_2^{(f)} \right)^2 \right] \\
& \times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right) \left(\sum_{f'} \bar{\psi}_2^{(f')} \psi_2^{(f')} \right) \right] \\
& \times \prod_f e^{\bar{\phi}_2^{(f)} \eta_1^{(f)}(n)} e^{-\phi_2^{(f)} \bar{\eta}_1^{(f)}(n-\hat{1})} e^{\bar{\phi}_1^{(f)} \bar{\zeta}_1^{(f)}(n-\hat{1})} e^{\phi_1^{(f)} \zeta_1^{(f)}(n)} \\
& \times e^{e^{\mu f/2} \bar{\chi}_2^{(f)} \eta_2^{(f)}(n)} e^{-e^{\mu f/2} \chi_2^{(f)} \bar{\eta}_2^{(f)}(n-\hat{2})} e^{-e^{\mu f/2} \bar{\chi}_1^{(f)} \bar{\zeta}_2^{(f)}(n-\hat{2})} e^{-e^{\mu f/2} \chi_1^{(f)} \zeta_2^{(f)}(n)}. \tag{3.6}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{T} &= \sum_{i_1, j_1, k_1, l_1} \sum_{i_2, j_2, k_2, l_2} \sum_{i'_1, j'_1, k'_1, l'_1} \sum_{i'_2, j'_2, k'_2, l'_2} \\
&\times (-1)^{i'_1 + i'_2 + k'_1 + k'_2} \\
&\times \exp \left[\frac{\mu_1}{2} (i_2 - j_2 + i'_2 - j'_2) + \frac{\mu_2}{2} (k_2 - l_2 + k'_2 - l'_2) \right] \\
&\times t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2} \zeta_1^{l_1} \zeta_2^{l_2} \bar{\eta}_1^{k'_1} \bar{\eta}_2^{k'_2} \eta_1^{k_1} \eta_2^{k_2} \bar{\zeta}_1^{l'_1} \bar{\zeta}_2^{l'_2} \beta_1^{j_1} \beta_2^{j_2} \bar{\alpha}_1^{i'_1} \bar{\alpha}_2^{i'_2} \alpha_1^{i_1} \alpha_2^{i_2} \bar{\beta}_1^{j'_1} \bar{\beta}_2^{j'_2} \\
&= \sum_{i_1, j_1, k_1, l_1} \sum_{i_2, j_2, k_2, l_2} \sum_{i'_1, j'_1, k'_1, l'_1} \sum_{i'_2, j'_2, k'_2, l'_2} \\
&\times (-1)^{i'_1 + i'_2 + k'_1 + k'_2} \\
&\times \exp \left[\frac{\mu_1}{2} (i_2 - j_2 + i'_2 - j'_2) + \frac{\mu_2}{2} (k_2 - l_2 + k'_2 - l'_2) \right] \\
&\times \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \left(\frac{1}{\sqrt{2}} \right)^{k_1 + l_1 + k_2 + l_2 + k'_1 + l'_1 + k'_2 + l'_2} \\
&\times (-1)^{P_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}} \\
&\times t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2} \alpha_1^{i_1} \beta_1^{j_1} \eta_1^{k_1} \zeta_1^{l_1} \alpha_2^{i_2} \beta_2^{j_2} \eta_2^{k_2} \zeta_2^{l_2} \bar{\zeta}_1^{l'_1} \bar{\eta}_1^{k'_1} \bar{\beta}_1^{j'_1} \bar{\alpha}_1^{i'_1} \bar{\zeta}_2^{l'_2} \bar{\eta}_2^{k'_2} \bar{\beta}_2^{j'_2} \bar{\alpha}_2^{i'_2}, \quad (3.7)
\end{aligned}$$

with

$$\begin{aligned}
P_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2} &= i_1(l_1 + l_2 + k'_1 + k'_2 + k_1 + k_2 + l'_1 + l'_2 + j_1 + j_2 + i'_1 + i'_2) \\
&\quad + j_1(l_1 + l_2 + k'_1 + k'_2 + k_1 + k_2 + l'_1 + l'_2) \\
&\quad + k_1(l_1 + l_2 + k'_1 + k'_2) \\
&\quad + i_2(l_2 + k'_1 + k'_2 + k_2 + l'_1 + l'_2 + j_2 + i'_1 + i'_2) \\
&\quad + j_2(l_2 + k'_1 + k'_2 + k_2 + l'_1 + l'_2) \\
&\quad + k_2(l_2 + k'_1 + k'_2) \\
&\quad + l'_1(k'_1 + k'_2) \\
&\quad + j'_1(k'_2 + l'_2 + i'_1 + i'_2) \\
&\quad + i'_1(k'_2 + l'_2) \\
&\quad + l'_2 k'_2 \\
&\quad + j'_2 i'_2. \quad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2} &= \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} \right) \\
&\times \exp \left[- \sum_{f, f'} \bar{\psi}_1^{(f)} \left\{ (M_f + 2) \delta_{ff'} + i h_a (\tau_a)^{ff'} \right\} \psi_1^{(f')} \right] \\
&\times \exp \left[- \sum_{f, f'} \bar{\psi}_2^{(f)} \left\{ (M_f + 2) \delta_{ff'} - i h_a (\tau_a)^{ff'} \right\} \psi_2^{(f')} \right] \\
&\times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{4} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right)^2 + \frac{g_\sigma^2 - g_\pi^2}{4} \left(\sum_f \bar{\psi}_2^{(f)} \psi_2^{(f)} \right)^2 \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right) \left(\sum_{f'} \bar{\psi}_2^{(f')} \psi_2^{(f')} \right) \right] \\
&\times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
&\times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1} \\
&= \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} \right) \\
&\times \exp \left[- \sum_{f, f'} \bar{\psi}_1^{(f)} \left\{ (M_f + 2) \delta_{ff'} + i h_a (\tau_a)^{ff'} \right\} \psi_1^{(f')} \right] \\
&\times \exp \left[- \sum_{f, f'} \bar{\psi}_2^{(f)} \left\{ (M_f + 2) \delta_{ff'} - i h_a (\tau_a)^{ff'} \right\} \psi_2^{(f')} \right] \\
&\times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{2} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} + \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \right) \right] \\
&\times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
&\times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1}, \\
&\hspace{15em} (3.9)
\end{aligned}$$

where we have renormalized original Grassmann variables by a factor $\sqrt{2}$. When $a = 3$, setting $h_3 = H$ and $\tau_3 = \sigma_z$,

$$\begin{aligned}
t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2} = & \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} \right) \\
& \times \exp \left[- (M_1 + 2 + iH) \bar{\psi}_1^{(1)} \psi_1^{(1)} - (M_1 + 2 - iH) \bar{\psi}_2^{(1)} \psi_2^{(1)} \right] \\
& \times \exp \left[- (M_2 + 2 - iH) \bar{\psi}_1^{(2)} \psi_1^{(2)} - (M_2 + 2 + iH) \bar{\psi}_2^{(2)} \psi_2^{(2)} \right] \\
& \times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{2} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \right) \right] \\
& \times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} + \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \right) \right] \\
& \times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{2} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \right) \right] \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1}.
\end{aligned} \tag{3.10}$$

¹ As we will see, there is a structure such that

$$t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2} = \sum_{\alpha=0,2,4,6,8} t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{[\alpha]}. \tag{3.11}$$

Note that

$$d\psi_1^{(1)} d\bar{\psi}_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\bar{\psi}_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_2^{(2)} = d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \tag{3.12}$$

holds. Our strategy is to evaluate the Grassmann integrals like

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1},
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \\
& \times \bar{\psi}_1^{(1)} \psi_1^{(1)} \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1},
\end{aligned} \tag{3.14}$$

¹The choice of τ_a just affects the bilinear terms. Since we have chosen flavor diagonal matrix σ_3 as τ_a , it is easy to obtain the tensor network representation with the choice of $\tau_a = \mathbb{1}$. This technique is useful when we evaluate $\langle \bar{\psi} i \gamma_5 \mathbb{1} \psi \rangle$. This is also the case with $N_f = 3$.

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \\
& \times \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1}, \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \\
& \times \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1}, \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \\
& \times \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1}. \tag{3.17}
\end{aligned}$$

Eq. (3.13) is evaluated as

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} d\psi_1^{(2)} d\psi_2^{(2)} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} \\
& \times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1} \\
& = d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} d\psi_1^{(1)} d\psi_2^{(1)} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} d\psi_1^{(2)} d\psi_2^{(2)} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1}, \tag{3.18}
\end{aligned}$$

which can be described as a product of the following tensors,

$$\bar{A}_{abcd}^{(f)} = \int \int d\bar{\psi}_1^{(f)} d\bar{\psi}_2^{(f)} (\bar{\chi}_1^{(f)})^a (\bar{\phi}_1^{(f)})^b (\bar{\chi}_2^{(f)})^c (\bar{\phi}_2^{(f)})^d, \tag{3.19}$$

$$A_{abcd}^{(f)} = \int \int d\psi_1^{(f)} d\psi_2^{(f)} (\chi_2^{(f)})^a (\phi_2^{(f)})^b (\chi_1^{(f)})^c (\phi_1^{(f)})^d. \tag{3.20}$$

² Eq. (3.14) is evaluated as

$$\begin{aligned}
& d\psi_1^{(1)} d\psi_2^{(1)} d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} \bar{\psi}_1^{(1)} \psi_1^{(1)} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times d\psi_1^{(2)} d\psi_2^{(2)} (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} d\bar{\psi}_1^{(2)} d\bar{\psi}_2^{(2)} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1} \\
& = d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} \bar{\psi}_1^{(1)} d\psi_1^{(1)} d\psi_2^{(1)} \psi_1^{(1)} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
& \times \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \\
& = P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \\
& \times d\bar{\psi}_1^{(1)} d\bar{\psi}_2^{(1)} (\bar{\psi}_2^{(1)})^{j'_2 + j'_1 + i_2 + i_1} \bar{\psi}_1^{(1)} d\psi_1^{(1)} d\psi_2^{(1)} \psi_1^{(1)} (\psi_2^{(1)})^{i'_2 + i'_1 + j_2 + j_1} \\
& = -P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)}. \tag{3.21}
\end{aligned}$$

where we set

$$P_{abcd} = (-1)^{a+b+c+d}, \tag{3.22}$$

$$I_{abcd} = (+i)^{a+b+c+d}. \tag{3.23}$$

The point is that when the integrand contains one extra fermion bilinear term, the sign flip occurs.

When the integrand contains two extra fermion bilinear terms, we need to care two patterns. When these two terms are coming from different flavors, as shown in Eq. (3.15), we acquire two extra minus signs as we explained previously. So no sign flip occurs. When these two terms are from the same flavor sector, however, we get minus sign because

$$d\psi_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_1^{(f)} d\bar{\psi}_2^{(f)} \bar{\psi}_1^{(f)} \psi_1^{(f)} \bar{\psi}_2^{(f)} \psi_2^{(f)} = -d\psi_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_1^{(f)} d\bar{\psi}_2^{(f)} \bar{\psi}_2^{(f)} \bar{\psi}_1^{(f)} \psi_2^{(f)} \psi_1^{(f)}. \tag{3.24}$$

These considerations make it clear that when we consider the integrals like in Eq. (3.16), we obtain no minus sign because we always encounter two negative signs which cancel each other. Eq. (3.17) also results in no extra negative sign by the same reason.

Let us explicitly obtain $t^{[\alpha]}$'s. Firstly, we have

$$t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{[0]} = \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)}. \tag{3.25}$$

Secondly, we consider $t^{[2]}$, which is constructed by

$$C_1^{[2]} \bar{\psi}_1^{(1)} \psi_1^{(1)} + C_2^{[2]} \bar{\psi}_2^{(1)} \psi_2^{(1)} + C_3^{[2]} \bar{\psi}_1^{(2)} \psi_1^{(2)} + C_4^{[2]} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.26}$$

with

$$C_1^{[2]} = -(M_1 + 2 + iH), \tag{3.27}$$

²The labels f in $A^{(f)}$ and $\bar{A}^{(f)}$ may not be necessary because we can identify them by their subscripts.

$$C_2^{[2]} = -(M_1 + 2 - iH), \quad (3.28)$$

$$C_3^{[2]} = -(M_2 + 2 - iH), \quad (3.29)$$

$$C_4^{[2]} = -(M_2 + 2 + iH). \quad (3.30)$$

As explained previously, when we integrate out Grassmann numbers we obtain extra minus sign.

$$\begin{aligned} t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{[2]} = \\ \left(-C_1^{[2]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} - C_2^{[2]} \right) \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \\ + \left(-C_3^{[2]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_4^{[2]} \right) \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1}. \end{aligned} \quad (3.31)$$

Next, we consider $t^{[4]}$ whose integrands contain

$$\begin{aligned} C_1^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + C_2^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + C_3^{[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \\ + C_4^{[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + C_5^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} + C_6^{[4]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \end{aligned} \quad (3.32)$$

with

$$C_1^{[4]} = (M_1 + 2 + iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \quad (3.33)$$

$$C_2^{[4]} = (M_1 + 2 + iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \quad (3.34)$$

$$C_3^{[4]} = (M_2 + 2 - iH) (M_1 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \quad (3.35)$$

$$C_4^{[4]} = (M_2 + 2 + iH) (M_1 + 2 - iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \quad (3.36)$$

$$C_5^{[4]} = (M_1 + 2 + iH) (M_1 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \quad (3.37)$$

$$C_6^{[4]} = (M_2 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}. \quad (3.38)$$

The last two lines are just constructed by the same flavor degree of freedom, we get minus signs. Consequently,

$$\begin{aligned}
& t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{[4]} \\
&= \left\{ C_1^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_2^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + C_3^{[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_4^{[4]} \right\} \\
&\times \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \\
&- C_5^{[4]} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \\
&- C_6^{[4]} \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0}.
\end{aligned} \tag{3.39}$$

Next task is to consider

$$\begin{aligned}
& C_1^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + C_2^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \\
& + C_3^{[6]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(1)} \psi_1^{(1)} + C_4^{[6]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(1)} \psi_2^{(1)},
\end{aligned} \tag{3.40}$$

with

$$\begin{aligned}
C_1^{[6]} &= - (M_1 + 2 + iH) (M_1 + 2 - iH) (M_2 + 2 - iH) \\
&- \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 + iH) - \frac{g_\sigma^2 - g_\pi^2}{2} (M_1 + 2 - iH) - \frac{g_\sigma^2 + g_\pi^2}{2} (M_2 + 2 - iH),
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
C_2^{[6]} &= - (M_1 + 2 + iH) (M_1 + 2 - iH) (M_2 + 2 + iH) \\
&- \frac{g_\sigma^2 - g_\pi^2}{2} (M_1 + 2 + iH) - \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 - iH) - \frac{g_\sigma^2 + g_\pi^2}{2} (M_2 + 2 + iH),
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
C_3^{[6]} &= - (M_1 + 2 + iH) (M_2 + 2 - iH) (M_2 + 2 + iH) \\
&- \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 + iH) - \frac{g_\sigma^2 + g_\pi^2}{2} (M_2 + 2 - iH) - \frac{g_\sigma^2 - g_\pi^2}{2} (M_2 + 2 + iH),
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
C_4^{[6]} &= - (M_1 + 2 - iH) (M_2 + 2 - iH) (M_2 + 2 + iH) \\
&- \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 - iH) - \frac{g_\sigma^2 - g_\pi^2}{2} (M_2 + 2 - iH) - \frac{g_\sigma^2 + g_\pi^2}{2} (M_2 + 2 + iH).
\end{aligned} \tag{3.44}$$

As a result, we obtain

$$\begin{aligned}
& t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{[6]} = \\
& \left[C_1^{[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_2^{[6]} \right] \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \\
& + \left[C_3^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + C_4^{[6]} \right] \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0}.
\end{aligned} \tag{3.45}$$

The last task is to pick up all the integrand proportional to

$$\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.46)$$

that is the resulting form is

$$t_{i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{[8]} = C^{[8]} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0}, \quad (3.47)$$

where

$$\begin{aligned} C^{[8]} = & (M_1 + 2 + iH) (M_1 + 2 - iH) (M_2 + 2 + iH) (M_2 + 2 - iH) \\ & + \left(\frac{g_\sigma^2 - g_\pi^2}{2} \right)^2 + 2 \left(\frac{g_\sigma^2 + g_\pi^2}{2} \right)^2 \\ & + \frac{g_\sigma^2 - g_\pi^2}{2} (M_1 + 2 - iH) (M_2 + 2 + iH) \\ & + \frac{g_\sigma^2 - g_\pi^2}{2} (M_1 + 2 + iH) (M_2 + 2 - iH) \\ & + \frac{g_\sigma^2 + g_\pi^2}{2} (M_2 + 2 - iH) (M_2 + 2 + iH) \\ & + \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 + iH) (M_1 + 2 - iH) \\ & + \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 - iH) (M_2 + 2 - iH) \\ & + \frac{g_\sigma^2 + g_\pi^2}{2} (M_1 + 2 + iH) (M_2 + 2 + iH). \end{aligned} \quad (3.48)$$

3.2 Impurity tensors for $\langle \bar{\psi}\psi \rangle$, $\langle \bar{\psi}i\gamma_5\psi \rangle$, and $\langle \bar{\psi}i\gamma_5\tau_3\psi \rangle$

These impurities introduce additional two Grassmann numbers. We set $s_\ell^{(f)}$ tensor for $\langle \bar{\psi}_\ell^{(f)}\psi_\ell^{(f)} \rangle$. Note that

$$s_\ell^{(f)} = \sum_{\alpha=2,4,6,8} s_\ell^{(f)[\alpha]}. \quad (3.49)$$

Impurity for $\langle \bar{\psi}\psi \rangle$ is given by

$$s = s_1^{(1)} + s_2^{(1)} + s_1^{(2)} + s_2^{(2)}. \quad (3.50)$$

Impurity for $\langle \bar{\psi}i\gamma_5\psi \rangle$ is given by

$$s = i s_1^{(1)} - i s_2^{(1)} + i s_1^{(2)} - i s_2^{(2)}. \quad (3.51)$$

Impurity for $\langle \bar{\psi}i\gamma_5\tau_3\psi \rangle$ is given by

$$s = i s_1^{(1)} - i s_2^{(1)} - i s_1^{(2)} + i s_2^{(2)}. \quad (3.52)$$

Let us explicitly obtain $s_\ell^{(f)[\alpha]}$, $s_\ell^{(f)[2]}$,s are immediately obtained as

$$s_{1;i_1j_1k_1l_1i_2j_2k_2l_2i'_1j'_1k'_1l'_1i'_2j'_2k'_2l'_2}^{(1)[2]} = -P_{i_2i_1i'_1j_2} I_{j'_2i_2i'_2j_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \bar{A}_{l'_2l'_1k_2k_1}^{(2)} A_{k'_2k'_1l_2l_1}^{(2)}, \quad (3.53)$$

$$s_{2;i_1j_1k_1l_1i_2j_2k_2l_2i'_1j'_1k'_1l'_1i'_2j'_2k'_2l'_2}^{(1)[2]} = -\delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \bar{A}_{l'_2l'_1k_2k_1}^{(2)} A_{k'_2k'_1l_2l_1}^{(2)}, \quad (3.54)$$

$$s_{1;i_1j_1k_1l_1i_2j_2k_2l_2i'_1j'_1k'_1l'_1i'_2j'_2k'_2l'_2}^{(2)[2]} = -P_{k_2k_1k'_1l_2} I_{l'_2k_2k'_2l_2} \bar{A}_{j'_2j'_1i_2i_1}^{(1)} A_{i'_2i'_1j_2j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1}, \quad (3.55)$$

$$s_{1;i_1j_1k_1l_1i_2j_2k_2l_2i'_1j'_1k'_1l'_1i'_2j'_2k'_2l'_2}^{(2)[2]} = -\bar{A}_{j'_2j'_1i_2i_1}^{(1)} A_{i'_2i'_1j_2j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1}. \quad (3.56)$$

$s_\ell^{(f)[4]}$,s acquire coefficients. For $s_1^{(1)[4]}$, we have

$$a_1^{(1)[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_1^{(1)[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_1^{(1)[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)}, \quad (3.57)$$

with

$$a_1^{(1)[4]} = -(M_2 + 2 - iH), \quad (3.58)$$

$$b_1^{(1)[4]} = -(M_2 + 2 + iH), \quad (3.59)$$

$$c_1^{(1)[4]} = -(M_1 + 2 - iH), \quad (3.60)$$

and as a result,

$$\begin{aligned} & s_{1;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(1)[4]} \\ &= \left(a_1^{(1)[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + b_1^{(1)[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \right) \\ & \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\ & - c_1^{(1)[4]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)}. \end{aligned} \quad (3.61)$$

For $s_2^{(1)[4]}$, we have

$$a_2^{(1)[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_2^{(1)[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_2^{(1)[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)}, \quad (3.62)$$

with

$$a_2^{(1)[4]} = -(M_2 + 2 - iH), \quad (3.63)$$

$$b_2^{(1)[4]} = -(M_2 + 2 + iH), \quad (3.64)$$

$$c_2^{(1)[4]} = -(M_1 + 2 + iH), \quad (3.65)$$

and as a result,

$$\begin{aligned} & s_{2;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(1)[4]} \\ &= \left(a_2^{(1)[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + b_2^{(1)[4]} \right) \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\ & - c_2^{(1)[4]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)}. \end{aligned} \quad (3.66)$$

For $s_1^{(2)[4]}$, we have

$$a_1^{(2)[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_1^{(2)[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + c_1^{(2)[4]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.67)$$

with

$$a_1^{(2)[4]} = -(M_1 + 2 + iH), \quad (3.68)$$

$$b_1^{(2)[4]} = -(M_1 + 2 - iH), \quad (3.69)$$

$$c_1^{(2)[4]} = -(M_2 + 2 + iH), \quad (3.70)$$

and as a result,

$$\begin{aligned}
& s_{1;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(2)[4]} \\
&= \left(a_1^{(2)[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + b_1^{(2)[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \right) \\
&\times \delta_{j_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\
&- c_1^{(2)[4]} \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}.
\end{aligned} \tag{3.71}$$

For $s_2^{(2)[4]}$, we have

$$a_2^{(2)[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + b_2^{(2)[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_2^{(2)[4]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.72}$$

with

$$a_2^{(2)[4]} = -(M_1 + 2 + iH), \tag{3.73}$$

$$b_2^{(2)[4]} = -(M_1 + 2 - iH), \tag{3.74}$$

$$c_2^{(2)[4]} = -(M_2 + 2 - iH), \tag{3.75}$$

and as a result,

$$\begin{aligned}
& s_{2;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(2)[4]} \\
&= \left(a_2^{(2)[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + b_2^{(2)[4]} \right) \delta_{j_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\
&- c_2^{(2)[4]} \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}.
\end{aligned} \tag{3.76}$$

Then we move on to consider $s_\ell^{(f)[6]}$'s. Let us consider $s_1^{(1)[6]}$, whose integrand contains

$$a_1^{(1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_1^{(1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_1^{(1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.77}$$

with

$$a_1^{(1)[6]} = (M_1 + 2 - iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.78}$$

$$b_1^{(1)[6]} = (M_1 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.79}$$

$$c_1^{(1)[6]} = (M_2 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \tag{3.80}$$

and

$$\begin{aligned}
& s_{1;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(1)[6]} = \\
& \left(a_1^{(1)[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + b_1^{(1)[6]} \right) \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\
& + c_1^{(1)[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}.
\end{aligned} \tag{3.81}$$

We next consider $s_2^{(1)[6]}$, whose integrand contains

$$a_2^{(1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_2^{(1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_2^{(1)[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.82}$$

with

$$a_2^{(1)[6]} = (M_1 + 2 + iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \tag{3.83}$$

$$b_2^{(1)[6]} = (M_1 + 2 + iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.84}$$

$$c_2^{(1)[6]} = (M_2 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.85}$$

and

$$\begin{aligned}
& s_{2;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(1)[6]} = \\
& \left(a_2^{(1)[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + b_2^{(1)[6]} \right) \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\
& + c_2^{(1)[6]} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}.
\end{aligned} \tag{3.86}$$

We consider $s_1^{(2)[6]}$, whose integrand contains

$$a_1^{(2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_1^{(2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_1^{(2)[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.87}$$

with

$$a_1^{(2)[6]} = (M_1 + 2 + iH) (M_1 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.88}$$

$$b_1^{(2)[6]} = (M_1 + 2 + iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.89}$$

$$c_1^{(2)[6]} = (M_1 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \tag{3.90}$$

and

$$\begin{aligned}
& s_{1;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(2)[6]} = \\
& a_1^{(2)[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\
& + \left(b_1^{(2)[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + c_1^{(2)[6]} \right) \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}.
\end{aligned} \tag{3.91}$$

$s_2^{(2)[6]}$ contains

$$a_2^{(2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + b_2^{(2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + c_2^{(2)[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.92}$$

with

$$a_2^{(2)[6]} = (M_1 + 2 + iH) (M_1 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.93}$$

$$b_2^{(2)[6]} = (M_1 + 2 + iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \tag{3.94}$$

$$c_2^{(2)[6]} = (M_1 + 2 - iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \tag{3.95}$$

and

$$\begin{aligned}
& s_{2;i_1 j_1 k_1 l_1 i_2 j_2 k_2 l_2 i'_1 j'_1 k'_1 l'_1 i'_2 j'_2 k'_2 l'_2}^{(2)[6]} = \\
& a_2^{(2)[6]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\
& + \left(b_2^{(2)[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + c_2^{(2)[6]} \right) \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}.
\end{aligned} \tag{3.96}$$

Finally, we consider $s_\ell^{(f)[8]}$'s. Since we have

$$a_1^{(1)[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \tag{3.97}$$

with

$$\begin{aligned}
a_1^{(1)[8]} = & - (M_1 + 2 - iH) (M_2 + 2 - iH) (M_2 + 2 + iH) \\
& - (M_1 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{2} - (M_2 + 2 - iH) \frac{g_\sigma^2 - g_\pi^2}{2} - (M_2 + 2 + iH) \frac{g_\sigma^2 + g_\pi^2}{2},
\end{aligned} \tag{3.98}$$

$$s_1^{(1)[8]} = a_1^{(1)[8]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \tag{3.99}$$

follows. Similarly,

$$a_2^{(1)[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.100)$$

with

$$\begin{aligned} a_2^{(1)[8]} = & - (M_1 + 2 + iH) (M_2 + 2 - iH) (M_2 + 2 + iH) \\ & - (M_1 + 2 + iH) \frac{g_\sigma^2 + g_\pi^2}{2} - (M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{2} - (M_2 + 2 + iH) \frac{g_\sigma^2 - g_\pi^2}{2}, \end{aligned} \quad (3.101)$$

$$s_2^{(1)[8]} = a_2^{(1)[8]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}, \quad (3.102)$$

and

$$a_1^{(2)[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.103)$$

with

$$\begin{aligned} a_1^{(2)[8]} = & - (M_1 + 2 + iH) (M_1 + 2 - iH) (M_2 + 2 + iH) \\ & - (M_1 + 2 + iH) \frac{g_\sigma^2 - g_\pi^2}{2} - (M_1 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{2} - (M_2 + 2 + iH) \frac{g_\sigma^2 + g_\pi^2}{2}, \end{aligned} \quad (3.104)$$

$$s_1^{(2)[8]} = a_1^{(2)[8]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}, \quad (3.105)$$

and

$$a_2^{(2)[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.106)$$

with

$$\begin{aligned} a_2^{(2)[8]} = & - (M_1 + 2 + iH) (M_1 + 2 - iH) (M_2 + 2 - iH) \\ & - (M_1 + 2 + iH) \frac{g_\sigma^2 + g_\pi^2}{2} - (M_1 + 2 - iH) \frac{g_\sigma^2 - g_\pi^2}{2} - (M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{2}, \end{aligned} \quad (3.107)$$

$$s_2^{(2)[8]} = a_2^{(2)[8]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}. \quad (3.108)$$

3.3 Impurity tensors for $\langle |\bar{\psi} i \gamma_5 \psi|^2 \rangle$ and $\langle |\bar{\psi} i \gamma_5 \tau_3 \psi|^2 \rangle$

These impurities introduce additional four Grassmann numbers. We set $s_{\ell_1, \ell_2}^{(f, f')}$ tensor for $\langle \bar{\psi}_\ell^{(f)} \psi_\ell^{(f)} \bar{\psi}_{\ell'}^{(f')} \psi_{\ell'}^{(f')} \rangle$. Note that

$$s_{\ell, \ell'}^{(f, f')} = \sum_{\alpha=4,6,8} s_{\ell, \ell'}^{(f, f')[\alpha]}. \quad (3.109)$$

Impurity for $\langle |\bar{\psi} i \gamma_5 \psi|^2 \rangle$ is given by

$$s = -2s_{1,2}^{(1,1)} + 2s_{1,1}^{(1,2)} - 2s_{1,2}^{(1,2)} - 2s_{2,1}^{(1,2)} + 2s_{2,2}^{(1,2)} - 2s_{1,2}^{(2,2)}. \quad (3.110)$$

Impurity for $\langle |\bar{\psi} i \gamma_5 \tau_3 \psi|^2 \rangle$ is given by

$$s = -2s_{1,2}^{(1,1)} - 2s_{1,1}^{(1,2)} + 2s_{1,2}^{(1,2)} + 2s_{2,1}^{(1,2)} - 2s_{2,2}^{(1,2)} - 2s_{1,2}^{(2,2)}. \quad (3.111)$$

$s_{\ell, \ell'}^{(f, f')[4]}$'s do not have extra coefficient, so they are immediately obtained as

$$s_{1,2}^{(1,1)[4]} = -\delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)}, \quad (3.112)$$

$$s_{1,1}^{(1,2)[4]} = P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1}, \quad (3.113)$$

$$s_{2,1}^{(1,2)[4]} = P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1}, \quad (3.114)$$

$$s_{1,2}^{(1,2)[4]} = P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1}, \quad (3.115)$$

$$s_{2,2}^{(1,2)[4]} = \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1}, \quad (3.116)$$

$$s_{1,2}^{(2,2)[4]} = -\bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}. \quad (3.117)$$

Next, we consider $s_{\ell, \ell'}^{(f, f')[6]}$'s. For $s_{1,2}^{(1,1)[6]}$, we need to think

$$a_{1,2}^{(1,1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_{1,2}^{(1,1)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.118)$$

where

$$a_{1,2}^{(1,1)[6]} = -(M_2 + 2 - iH), \quad (3.119)$$

$$b_{1,2}^{(1,1)[6]} = -(M_2 + 2 + iH) \quad (3.120)$$

and

$$s_{1,2}^{(1,1)[6]} = \left(a_{1,2}^{(1,1)[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + b_{1,2}^{(1,1)[6]} \right) \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1}. \quad (3.121)$$

For $s_{1,1}^{(1,2)[6]}$, we need to think

$$a_{1,1}^{(1,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_{1,1}^{(1,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.122)$$

where

$$a_{1,1}^{(1,2)[6]} = -(M_1 + 2 - iH), \quad (3.123)$$

$$b_{1,1}^{(1,2)[6]} = -(M_2 + 2 + iH) \quad (3.124)$$

and

$$\begin{aligned} s_{1,1}^{(1,2)[6]} &= a_{1,1}^{(1,2)[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \\ &+ b_{1,1}^{(1,2)[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0}. \end{aligned} \quad (3.125)$$

For $s_{2,1}^{(1,2)[6]}$, we need to think

$$a_{2,1}^{(1,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + b_{2,1}^{(1,2)[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.126)$$

where

$$a_{2,1}^{(1,2)[6]} = -(M_1 + 2 + iH), \quad (3.127)$$

$$b_{2,1}^{(1,2)[6]} = -(M_2 + 2 + iH) \quad (3.128)$$

and

$$\begin{aligned} s_{2,1}^{(1,2)[6]} &= a_{2,1}^{(1,2)[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \\ &+ b_{2,1}^{(1,1)[6]} \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0}. \end{aligned} \quad (3.129)$$

For $s_{1,2}^{(1,2)[6]}$, we need to think

$$a_{1,2}^{(1,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + b_{1,2}^{(1,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.130)$$

where

$$a_{1,2}^{(1,2)[6]} = -(M_1 + 2 - iH), \quad (3.131)$$

$$b_{1,2}^{(1,2)[6]} = -(M_2 + 2 - iH) \quad (3.132)$$

and

$$\begin{aligned} s_{1,2}^{(1,2)[6]} &= a_{1,2}^{(1,2)[6]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\ &+ b_{1,2}^{(1,1)[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}. \end{aligned} \quad (3.133)$$

For $s_{2,2}^{(1,2)[6]}$, we need to think

$$a_{2,2}^{(1,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + b_{2,2}^{(1,2)[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.134)$$

where

$$a_{2,2}^{(1,2)[6]} = -(M_1 + 2 + iH), \quad (3.135)$$

$$b_{2,2}^{(1,2)[6]} = -(M_2 + 2 - iH) \quad (3.136)$$

and

$$\begin{aligned} s_{2,2}^{(1,2)[6]} &= a_{2,2}^{(1,2)[6]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \\ &+ b_{2,2}^{(1,1)[6]} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}. \end{aligned} \quad (3.137)$$

For $s_{1,2}^{(2,2)[6]}$, we need to think

$$a_{1,2}^{(2,2)[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + b_{1,2}^{(2,2)[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)}, \quad (3.138)$$

where

$$a_{1,2}^{(2,2)[6]} = -(M_1 + 2 + iH), \quad (3.139)$$

$$b_{1,2}^{(2,2)[6]} = -(M_1 + 2 - iH) \quad (3.140)$$

and

$$\begin{aligned} s_{1,2}^{(2,2)[6]} &= \left(a_{1,2}^{(2,1)[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + b_{1,2}^{(2,1)[6]} \right) \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}. \end{aligned} \quad (3.141)$$

Finally, we have

$$s_{\ell,\ell'}^{(f,f')[8]} = a_{\ell,\ell'}^{(f,f')[8]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0}, \quad (3.142)$$

where

$$s_{1,2}^{(1,1)[8]} = (M_2 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \quad (3.143)$$

$$s_{1,1}^{(1,2)[8]} = (M_1 + 2 - iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \quad (3.144)$$

$$s_{2,1}^{(1,2)[8]} = (M_1 + 2 + iH) (M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \quad (3.145)$$

$$s_{1,2}^{(1,2)[8]} = (M_1 + 2 - iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}, \quad (3.146)$$

$$s_{2,2}^{(1,2)[8]} = (M_1 + 2 + iH) (M_2 + 2 - iH) + \frac{g_\sigma^2 - g_\pi^2}{2}, \quad (3.147)$$

$$s_{1,2}^{(2,2)[8]} = (M_1 + 2 + iH) (M_1 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{2}. \quad (3.148)$$

4 $N_f = 3$

4.1 Pure tensor

Our Grassmann tensor is defined via

$$\begin{aligned}
\mathcal{T} = & \sum_{i_1, j_1, k_1, l_1, m_1, n_1} \sum_{i_2, j_2, k_2, l_2, m_2, n_2} \sum_{i'_1, j'_1, k'_1, l'_1, m'_1, n'_1} \sum_{i'_2, j'_2, k'_2, l'_2, m'_2, n'_2} \\
& \times (-1)^{i'_1 + i'_2 + k'_1 + k'_2 + m'_1 + m'_2} \\
& \times \exp \left[\frac{\mu_1}{2} (i_2 - j_2 + i'_2 - j'_2) + \frac{\mu_2}{2} (k_2 - l_2 + k'_2 - l'_2) + \frac{\mu_3}{2} (m_2 - n_2 + m'_2 - n'_2) \right] \\
& \times t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} \\
& \times \sigma_1^{n_1} \sigma_2^{n_2} \bar{\rho}_1^{m_1} \bar{\rho}_2^{m_2} \rho_1^{m_1} \rho_2^{m_2} \bar{\sigma}_1^{n_1} \bar{\sigma}_2^{n_2} \zeta_1^{l_1} \zeta_2^{l_2} \bar{\eta}_1^{k_1} \bar{\eta}_2^{k_2} \eta_1^{k_1} \eta_2^{k_2} \bar{\zeta}_1^{l'_1} \bar{\zeta}_2^{l'_2} \beta_1^{j_1} \beta_2^{j_2} \bar{\alpha}_1^{i_1} \bar{\alpha}_2^{i_2} \alpha_1^{i_1} \alpha_2^{i_2} \bar{\beta}_1^{j'_1} \bar{\beta}_2^{j'_2} \\
= & \sum_{i_1, j_1, k_1, l_1, m_1, n_1} \sum_{i_2, j_2, k_2, l_2, m_2, n_2} \sum_{i'_1, j'_1, k'_1, l'_1, m'_1, n'_1} \sum_{i'_2, j'_2, k'_2, l'_2, m'_2, n'_2} \\
& \times (-1)^{i'_1 + i'_2 + k'_1 + k'_2 + m'_1 + m'_2} \\
& \times \exp \left[\frac{\mu_1}{2} (i_2 - j_2 + i'_2 - j'_2) + \frac{\mu_2}{2} (k_2 - l_2 + k'_2 - l'_2) + \frac{\mu_3}{2} (m_2 - n_2 + m'_2 - n'_2) \right] \\
& \times \left(\frac{1}{\sqrt{2}} \right)^{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2} \left(\frac{1}{\sqrt{2}} \right)^{k_1 + l_1 + k_2 + l_2 + k'_1 + l'_1 + k'_2 + l'_2} \left(\frac{1}{\sqrt{2}} \right)^{m_1 + n_1 + m_2 + n_2 + m'_1 + n'_1 + m'_2 + n'_2} \\
& \times (-1)^{P_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2}} \\
& \times t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} \\
& \times \alpha_1^{i_1} \beta_1^{j_1} \eta_1^{k_1} \zeta_1^{l_1} \rho_1^{m_1} \sigma_1^{n_1} \alpha_2^{i_2} \beta_2^{j_2} \eta_2^{k_2} \zeta_2^{l_2} \rho_2^{m_2} \sigma_2^{n_2} \bar{\sigma}_1^{n'_1} \bar{\rho}_1^{m'_1} \bar{\zeta}_1^{l'_1} \bar{\eta}_1^{k'_1} \bar{\beta}_1^{j'_1} \bar{\alpha}_1^{i'_1} \bar{\sigma}_2^{n'_2} \bar{\rho}_2^{m'_2} \bar{\zeta}_2^{l'_2} \bar{\eta}_2^{k'_2} \bar{\beta}_2^{j'_2} \bar{\alpha}_2^{i'_2}, \quad (4.1)
\end{aligned}$$

with

$$\begin{aligned}
& P_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} = \\
& i_1(n_1 + n_2 + m'_1 + m'_2 + m_1 + m_2 + n'_1 + n'_2 + l_1 + l_2 + k'_1 + k'_2 + k_1 + k_2 + l'_1 + l'_2 + j_1 + j_2 + i'_1 + i'_2) \\
& + j_1(n_1 + n_2 + m'_1 + m'_2 + m_1 + m_2 + n'_1 + n'_2 + l_1 + l_2 + k'_1 + k'_2 + k_1 + k_2 + l'_1 + l'_2) \\
& + k_1(n_1 + n_2 + m'_1 + m'_2 + m_1 + m_2 + n'_1 + n'_2 + l_1 + l_2 + k'_1 + k'_2) \\
& + l_1(n_1 + n_2 + m'_1 + m'_2 + m_1 + m_2 + n'_1 + n'_2) \\
& + m_1(n_1 + n_2 + m'_1 + m'_2) \\
& + i_2(n_2 + m'_1 + m'_2 + m_2 + n'_1 + n'_2 + l_2 + k'_1 + k'_2 + k_2 + l'_1 + l'_2 + j_2 + i'_1 + i'_2) \\
& + j_2(n_2 + m'_1 + m'_2 + m_2 + n'_1 + n'_2 + l_2 + k'_1 + k'_2 + k_2 + l'_1 + l'_2) \\
& + k_2(n_2 + m'_1 + m'_2 + m_2 + n'_1 + n'_2 + l_2 + k'_1 + k'_2) \\
& + l_2(n_2 + m'_1 + m'_2 + m_2 + n'_1 + n'_2) \\
& + m_2(n_2 + m'_1 + m'_2) \\
& + n'_1(m'_1 + m'_2) \\
& + l'_1(m'_2 + n'_2 + k'_1 + k'_2) \\
& + k'_1(m'_2 + n'_2) \\
& + j'_1(m'_2 + n'_2 + k'_2 + l'_2 + i'_1 + i'_2) \\
& + i'_1(m'_2 + n'_2 + k'_2 + l'_2) \\
& + n'_2 m'_2 \\
& + l'_2 k'_2 \\
& + j'_2 i'_2
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
& t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} \\
&= \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} \right) \\
&\times \exp \left[- \sum_{f, f'} \bar{\psi}_1^{(f)} \left\{ (M_f + 2) \delta_{ff'} + i h_a(\tau_a)^{ff'} \right\} \psi_1^{(f')} \right] \\
&\times \exp \left[- \sum_{f, f'} \bar{\psi}_2^{(f)} \left\{ (M_f + 2) \delta_{ff'} - i h_a(\tau_a)^{ff'} \right\} \psi_2^{(f')} \right] \\
&\times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{6} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right)^2 + \frac{g_\sigma^2 - g_\pi^2}{6} \left(\sum_f \bar{\psi}_2^{(f)} \psi_2^{(f)} \right)^2 \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{3} \left(\sum_f \bar{\psi}_1^{(f)} \psi_1^{(f)} \right) \left(\sum_{f'} \bar{\psi}_2^{(f')} \psi_2^{(f')} \right) \right] \\
&\times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
&\times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1} \\
&\times (\bar{\chi}_1^{(3)})^{n'_2} (\bar{\phi}_1^{(3)})^{n'_1} (\bar{\chi}_2^{(3)})^{m_2} (\bar{\phi}_2^{(3)})^{m_1} (\chi_2^{(3)})^{m'_2} (\phi_2^{(3)})^{m'_1} (\chi_1^{(3)})^{n_2} (\phi_1^{(3)})^{n_1}. \tag{4.3}
\end{aligned}$$

Note that we renormalized original Grassmann numbers as in the case of $N_f = 2$. Choosing $a = 8$, we set $h_8 = H$ and $\tau_3 = \lambda_8$. Normalizing $\mathbf{H} = \mathbf{H}/\sqrt{3}$, we have

$$\begin{aligned}
& t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} \\
&= \left(\prod_f \int \int \int \int d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} \right) \\
&\times \exp \left[- (M_1 + 2 + iH) \bar{\psi}_1^{(1)} \psi_1^{(1)} - (M_1 + 2 - iH) \bar{\psi}_2^{(1)} \psi_2^{(1)} \right] \\
&\times \exp \left[- (M_2 + 2 + iH) \bar{\psi}_1^{(2)} \psi_1^{(2)} - (M_2 + 2 - iH) \bar{\psi}_2^{(2)} \psi_2^{(2)} \right] \\
&\times \exp \left[- (M_3 + 2 - 2iH) \bar{\psi}_1^{(3)} \psi_1^{(3)} - (M_3 + 2 + 2iH) \bar{\psi}_2^{(3)} \psi_2^{(3)} \right] \\
&\times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{3} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 - g_\pi^2}{3} \left(\bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{3} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} + \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{3} \left(\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{3} \left(\bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \right) \right] \\
&\times \exp \left[\frac{g_\sigma^2 + g_\pi^2}{3} \left(\bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \right) \right] \\
&\times (\bar{\chi}_1^{(1)})^{j'_2} (\bar{\phi}_1^{(1)})^{j'_1} (\bar{\chi}_2^{(1)})^{i_2} (\bar{\phi}_2^{(1)})^{i_1} (\chi_2^{(1)})^{i'_2} (\phi_2^{(1)})^{i'_1} (\chi_1^{(1)})^{j_2} (\phi_1^{(1)})^{j_1} \\
&\times (\bar{\chi}_1^{(2)})^{l'_2} (\bar{\phi}_1^{(2)})^{l'_1} (\bar{\chi}_2^{(2)})^{k_2} (\bar{\phi}_2^{(2)})^{k_1} (\chi_2^{(2)})^{k'_2} (\phi_2^{(2)})^{k'_1} (\chi_1^{(2)})^{l_2} (\phi_1^{(2)})^{l_1} \\
&\times (\bar{\chi}_1^{(3)})^{n'_2} (\bar{\phi}_1^{(3)})^{n'_1} (\bar{\chi}_2^{(3)})^{m_2} (\bar{\phi}_2^{(3)})^{m_1} (\chi_2^{(3)})^{m'_2} (\phi_2^{(3)})^{m'_1} (\chi_1^{(3)})^{n_2} (\phi_1^{(3)})^{n_1}.
\end{aligned} \tag{4.4}$$

With $N_f = 3$ the above tensor is decomposed into

$$t = \sum_{\alpha=0,2,4,6,8,10,12} t^{[\alpha]}. \tag{4.5}$$

³ Since

$$\prod_{f=1}^3 d\psi_1^{(f)} d\bar{\psi}_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_2^{(f)} = (-1)^3 \prod_{f=1}^3 d\psi_1^{(f)} d\psi_2^{(f)} d\bar{\psi}_1^{(f)} d\bar{\psi}_2^{(f)} \tag{4.6}$$

holds, there is an overall negative sign. For a while, we consider $-t$ instead of t to neglect this overall sign.

³Numbers of terms in $t^{[\alpha]}$ are equal to $N_f \times N C_{\alpha/2}$ with N the number of elements in fermion fields.

When there is no additional fermion bilinear term, the integral is described as a product of $A^{(f)}$ and $\bar{A}^{(f)}$. If we have one extra bilinear term, the sign flips. If we have two bilinear terms which are from different flavor sectors, two negative signs cancel out. However, when they are from the same flavor sector, we get a minus sign. If we have three bilinear terms, there are two possibilities. One is like

$$\bar{\psi}_1^{(1)}\psi_1^{(1)}\bar{\psi}_2^{(2)}\psi_2^{(2)}\bar{\psi}_1^{(3)}\psi_1^{(3)}, \quad (4.7)$$

where we obtain the factor $(-1)^3$ as a result of the integration. The other is like

$$\bar{\psi}_1^{(1)}\psi_1^{(1)}\bar{\psi}_2^{(1)}\psi_2^{(1)}\bar{\psi}_1^{(3)}\psi_1^{(3)}, \quad (4.8)$$

where we obtain the factor $(-1)^2$, so no sign flip occurs. Eq. (4.7) is a new possibility arising with $N_f = 3$. With four bilinear terms, we have patterns. For instance,

$$\bar{\psi}_1^{(1)}\psi_1^{(1)}\bar{\psi}_2^{(1)}\psi_2^{(1)}\bar{\psi}_1^{(2)}\psi_1^{(2)}\bar{\psi}_1^{(3)}\psi_1^{(3)}, \quad (4.9)$$

and we get $(-1)^3$. When we have

$$\bar{\psi}_1^{(1)}\psi_1^{(1)}\bar{\psi}_2^{(1)}\psi_2^{(1)}\bar{\psi}_1^{(2)}\psi_1^{(2)}\bar{\psi}_2^{(2)}\psi_2^{(2)}, \quad (4.10)$$

the resulting factor is $(-1)^2$. These are also new patterns in $N_f = 3$. When we have five bilinear factors, we encounter

$$\bar{\psi}_1^{(1)}\psi_1^{(1)}\bar{\psi}_2^{(1)}\psi_2^{(1)}\bar{\psi}_1^{(2)}\psi_1^{(2)}\bar{\psi}_2^{(2)}\psi_2^{(2)}\bar{\psi}_1^{(3)}\psi_1^{(3)}, \quad (4.11)$$

for example, which gives us $(-1)^3$, so the sign must flip. This is also the case with six bilinear terms.

Let us explicitly determine $-t$. For $\alpha = 0$, we have

$$-t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2}^{[0]} = \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)}. \quad (4.12)$$

Secondly, we evaluate $-t^{[2]}$, which is established by

$$C_1^{[2]}\bar{\psi}_1^{(1)}\psi_1^{(1)} + C_2^{[2]}\bar{\psi}_2^{(1)}\psi_2^{(1)} + C_3^{[2]}\bar{\psi}_1^{(2)}\psi_1^{(2)} + C_4^{[2]}\bar{\psi}_2^{(2)}\psi_2^{(2)} + C_5^{[2]}\bar{\psi}_1^{(3)}\psi_1^{(3)} + C_6^{[2]}\bar{\psi}_2^{(3)}\psi_2^{(3)}, \quad (4.13)$$

where $C^{[2]}$'s are listed in Table 1. As discussed above, there must be a minus sign. Therefore,

$$\begin{aligned} -t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2}^{[2]} = & \left(-C_1^{[2]} P_{i_2 i_1 i'_1 j'_1} I_{j'_2 i_2 i'_2 j_2} - C_2^{[2]} \right) \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\ & + \left(-C_3^{[2]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_4^{[2]} \right) \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\ & + \left(-C_5^{[2]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_6^{[2]} \right) \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \delta_{n'_2 + n'_1 + m_2 + m_1, 1} \delta_{m'_2 + m'_1 + n_2 + n_1, 1}. \end{aligned} \quad (4.14)$$

Table 1: List of $C^{[2]}$'s.

$C_1^{[2]}$	$-(M_1 + 2 + iH)$
$C_2^{[2]}$	$-(M_1 + 2 - iH)$
$C_3^{[2]}$	$-(M_2 + 2 + iH)$
$C_4^{[2]}$	$-(M_2 + 2 - iH)$
$C_5^{[2]}$	$-(M_3 + 2 - 2iH)$
$C_6^{[2]}$	$-(M_3 + 2 + 2iH)$

For $\alpha = 4$, we have a product of two bilinear terms such that

$$\begin{aligned}
& C_1^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + C_2^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + C_3^{[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + C_4^{[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \\
& + C_5^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_6^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_7^{[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_8^{[4]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_9^{[4]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_{10}^{[4]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_{11}^{[4]} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_{12}^{[4]} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{13}^{[4]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} + C_{14}^{[4]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + C_{15}^{[4]} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)}. \tag{4.15}
\end{aligned}$$

$C^{[4]}$'s are listed in Table 2. The last three terms are constructed from the same flavor component, so we have additional minus sign. Therefore, the Grassmann integral gives us

$$\begin{aligned}
& -t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} \\
& = \left(C_1^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_2^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + C_3^{[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_4^{[4]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& + \left(C_5^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_6^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + C_7^{[4]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_8^{[4]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \delta_{n'_2 + n'_1 + m_2 + m_1, 1} \delta_{m'_2 + m'_1 + n_2 + n_1, 1} \\
& + \left(C_9^{[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_{10}^{[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_{11}^{[4]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_{12}^{[4]} \right) \\
& \times \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \delta_{n'_2 + n'_1 + m_2 + m_1, 1} \delta_{m'_2 + m'_1 + n_2 + n_1, 1} \\
& - C_{13}^{[4]} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& - C_{14}^{[4]} \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& - C_{15}^{[4]} \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0}. \tag{4.16}
\end{aligned}$$

Table 2: List of $C^{[4]}$'s.

$C_1^{[4]}$	$(M_1 + 2 + iH)(M_2 + 2 + iH) + \frac{g_\sigma^2 - g_\pi^2}{3}$
$C_2^{[4]}$	$(M_1 + 2 + iH)(M_2 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_3^{[4]}$	$(M_1 + 2 - iH)(M_2 + 2 + iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_4^{[4]}$	$(M_1 + 2 - iH)(M_2 + 2 - iH) + \frac{g_\sigma^2 - g_\pi^2}{3}$
$C_5^{[4]}$	$(M_1 + 2 + iH)(M_3 + 2 - 2iH) + \frac{g_\sigma^2 - g_\pi^2}{3}$
$C_6^{[4]}$	$(M_1 + 2 + iH)(M_3 + 2 + 2iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_7^{[4]}$	$(M_1 + 2 - iH)(M_3 + 2 - 2iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_8^{[4]}$	$(M_1 + 2 - iH)(M_3 + 2 + 2iH) + \frac{g_\sigma^2 - g_\pi^2}{3}$
$C_9^{[4]}$	$(M_2 + 2 + iH)(M_3 + 2 - 2iH) + \frac{g_\sigma^2 - g_\pi^2}{3}$
$C_{10}^{[4]}$	$(M_2 + 2 + iH)(M_3 + 2 + 2iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_{11}^{[4]}$	$(M_2 + 2 - iH)(M_3 + 2 - 2iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_{12}^{[4]}$	$(M_2 + 2 - iH)(M_3 + 2 + 2iH) + \frac{g_\sigma^2 - g_\pi^2}{3}$
$C_{13}^{[4]}$	$(M_1 + 2 + iH)(M_1 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_{14}^{[4]}$	$(M_2 + 2 + iH)(M_2 + 2 - iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$
$C_{15}^{[4]}$	$(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) + \frac{g_\sigma^2 + g_\pi^2}{3}$

For $\alpha = 6$, there are 20 terms which contain three bilinear factors:

$$\begin{aligned}
& C_1^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_2^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \\
& + C_3^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_4^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_5^{[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_6^{[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \\
& + C_7^{[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_8^{[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_9^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} + C_{10}^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \\
& + C_{11}^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_{12}^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{13}^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} + C_{14}^{[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \\
& + C_{15}^{[6]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_{16}^{[6]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{17}^{[6]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_{18}^{[6]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{19}^{[6]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_{20}^{[6]} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)}. \tag{4.17}
\end{aligned}$$

$C^{[6]}$'s are given as

$$\begin{aligned}
C_1^{[6]} = & -(M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \\
& - (M_1 + 2 + iH) \frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 + iH) \frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3}, \tag{4.18}
\end{aligned}$$

$$C_2^{[6]} = -(M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.19)$$

$$C_3^{[6]} = -(M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3}, \quad (4.20)$$

$$C_4^{[6]} = -(M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.21)$$

$$C_5^{[6]} = -(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \\ - (M_1 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.22)$$

$$C_6^{[6]} = -(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\ - (M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3}, \quad (4.23)$$

$$C_7^{[6]} = -(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.24)$$

$$C_8^{[6]} = -(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3}, \quad (4.25)$$

$$C_9^{[6]} = -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_1 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.26)$$

$$C_{10}^{[6]} = -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.27)$$

$$C_{11}^{[6]} = -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 - 2iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_1 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.28)$$

$$C_{12}^{[6]} = -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.29)$$

$$C_{13}^{[6]} = -(M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3}, \quad (4.30)$$

$$C_{14}^{[6]} = -(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \\ - (M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.31)$$

$$C_{15}^{[6]} = -(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\ - (M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.32)$$

$$C_{16}^{[6]} = -(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\ - (M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.33)$$

$$C_{17}^{[6]} = -(M_1 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3}, \quad (4.34)$$

$$C_{18}^{[6]} = -(M_1 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\ - (M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}, \quad (4.35)$$

$$C_{19}^{[6]} = -(M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\ - (M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3}, \quad (4.36)$$

$$C_{20}^{[6]} = -(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\ - (M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} - (M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} - (M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3}. \quad (4.37)$$

By integrating out the Grassmann fields, we obtain

$$\begin{aligned}
& -t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2} \\
& = \left(-C_1^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_2^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} \right. \\
& - C_3^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_4^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} - C_5^{[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} \\
& \left. - C_6^{[6]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_7^{[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_8^{[6]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& + \left(C_9^{[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_{10}^{[6]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& + \left(C_{11}^{[6]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_{12}^{[6]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& + \left(C_{13}^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + C_{14}^{[6]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& + \left(C_{15}^{[6]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_{16}^{[6]} \right) \\
& \times \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& + \left(C_{17}^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} + C_{18}^{[6]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \delta_{n'_2+n'_1+m_2+m_1,0} \delta_{m'_2+m'_1+n_2+n_1,0} \\
& + \left(C_{19}^{[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} + C_{20}^{[6]} \right) \\
& \times \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \delta_{n'_2+n'_1+m_2+m_1,0} \delta_{m'_2+m'_1+n_2+n_1,0}. \tag{4.38}
\end{aligned}$$

For $\alpha = 8$, we have

$$\begin{aligned}
& C_1^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_2^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \\
& + C_3^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_4^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_5^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} + C_6^{[8]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \\
& + C_7^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_8^{[8]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_9^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_{10}^{[8]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{11}^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} + C_{12}^{[8]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{13}^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \\
& + C_{14}^{[8]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_{15}^{[8]} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)}. \tag{4.39}
\end{aligned}$$

We obtain

$$\begin{aligned}
C_1^{[8]} &= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 3\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)\left(\frac{g_\sigma^2 - g_\pi^2}{3}\right), \tag{4.40}
\end{aligned}$$

$$\begin{aligned}
C_2^{[8]} &= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.41}
\end{aligned}$$

$$\begin{aligned}
C_3^{[8]} &= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.42}
\end{aligned}$$

$$\begin{aligned}
C_4^{[8]} &= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 3\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)\left(\frac{g_\sigma^2 - g_\pi^2}{3}\right), \tag{4.43}
\end{aligned}$$

$$\begin{aligned}
C_5^{[8]} &= (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ 3\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)\left(\frac{g_\sigma^2 - g_\pi^2}{3}\right), \tag{4.44}
\end{aligned}$$

$$\begin{aligned}
C_6^{[8]} &= (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\
&+ (M_1 + 2 - iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.45}
\end{aligned}$$

$$\begin{aligned}
C_7^{[8]} &= (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.46}
\end{aligned}$$

$$\begin{aligned}
C_8^{[8]} &= (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 - iH)(M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 3\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)\left(\frac{g_\sigma^2 - g_\pi^2}{3}\right), \tag{4.47}
\end{aligned}$$

$$\begin{aligned}
C_9^{[8]} &= (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ 3\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)\left(\frac{g_\sigma^2 - g_\pi^2}{3}\right), \tag{4.48}
\end{aligned}$$

$$\begin{aligned}
C_{10}^{[8]} &= (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 - iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.49}
\end{aligned}$$

$$\begin{aligned}
C_{11}^{[8]} &= (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_2 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.50}
\end{aligned}$$

$$\begin{aligned}
C_{12}^{[8]} &= (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 - iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_2 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ 3\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)\left(\frac{g_\sigma^2 - g_\pi^2}{3}\right), \tag{4.51}
\end{aligned}$$

$$\begin{aligned}
C_{13}^{[8]} &= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_2 + 2 + iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_2 + 2 - iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_2 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.52}
\end{aligned}$$

$$\begin{aligned}
C_{14}^{[8]} &= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_1 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2, \tag{4.53}
\end{aligned}$$

$$\begin{aligned}
C_{15}^{[8]} &= (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_2 + 2 + iH)(M_2 + 2 - iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 + iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} + (M_2 + 2 - iH)(M_3 + 2 - 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_2 + 2 - iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 - g_\pi^2}{3} + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH)\frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ 2\left(\frac{g_\sigma^2 + g_\pi^2}{3}\right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3}\right)^2. \tag{4.54}
\end{aligned}$$

Integrating out the original Grassmann variables, one finds

$$\begin{aligned}
& -t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2}^{[8]} = \\
& \left(-C_1^{[8]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_2^{[8]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_3^{[8]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_4^{[8]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \delta_{n'_2 + n'_1 + m_2 + m_1, 1} \delta_{m'_2 + m'_1 + n_2 + n_1, 1} \\
& + \left(-C_5^{[8]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_6^{[8]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_7^{[8]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} - C_8^{[8]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \delta_{n'_2 + n'_1 + m_2 + m_1, 1} \delta_{m'_2 + m'_1 + n_2 + n_1, 1} \\
& + \left(-C_9^{[8]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_{10}^{[8]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_{11}^{[8]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} - C_{12}^{[8]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0} \\
& + C_{13}^{[8]} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& + C_{14}^{[8]} \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \bar{A}_{l'_2 l'_1 k_2 k_1}^{(2)} A_{k'_2 k'_1 l_2 l_1}^{(2)} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0} \\
& + C_{15}^{[8]} \bar{A}_{j'_2 j'_1 i_2 i_1}^{(1)} A_{i'_2 i'_1 j_2 j_1}^{(1)} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0}. \tag{4.55}
\end{aligned}$$

For $\alpha = 10$, we have 6 terms such that

$$\begin{aligned}
& C_1^{[10]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \\
& + C_2^{[10]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_3^{[10]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_4^{[10]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_5^{[10]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)} \\
& + C_6^{[10]} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)}, \tag{4.56}
\end{aligned}$$

where

$$\begin{aligned}
C_1^{[10]} = & -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_1 + 2 - iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_2 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_2 + 2 - iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_3 + 2 - 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right], \tag{4.57}
\end{aligned}$$

$$\begin{aligned}
C_2^{[10]} = & -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_1 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_2 + 2 + iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_2 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_3 + 2 + 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right), \tag{4.58}
\end{aligned}$$

$$\begin{aligned}
C_3^{[10]} = & -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 - 3iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_1 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_2 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_3 + 2 - 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_3 + 2 + 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right], \tag{4.59}
\end{aligned}$$

$$\begin{aligned}
C_4^{[10]} = & -(M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 - 3iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_1 + 2 - iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_2 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_3 + 2 - 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_3 + 2 + 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right), \tag{4.60}
\end{aligned}$$

$$\begin{aligned}
C_5^{[10]} = & -(M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 - 3iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 + iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_2 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_2 + 2 - iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_3 + 2 - 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_3 + 2 + 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right], \tag{4.61}
\end{aligned}$$

$$\begin{aligned}
C_6^{[10]} = & -(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 3iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_1 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
& - (M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
& - (M_1 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_2 + 2 + iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_2 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& - (M_3 + 2 - 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& - (M_3 + 2 + 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right].
\end{aligned} \tag{4.62}$$

The integration gives us

$$\begin{aligned}
& -t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2}^{[10]} = \\
& \left(-C_1^{[10]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} - C_2^{[10]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \delta_{n'_2 + n'_1 + m_2 + m_1, 1} \delta_{m'_2 + m'_1 + n_2 + n_1, 1} \\
& \left(-C_3^{[10]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} - C_4^{[10]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 1} \delta_{k'_2 + k'_1 + l_2 + l_1, 1} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0} \\
& \left(-C_5^{[10]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} - C_6^{[10]} \right) \\
& \times \delta_{j'_2 + j'_1 + i_2 + i_1, 1} \delta_{i'_2 + i'_1 + j_2 + j_1, 1} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0}.
\end{aligned} \tag{4.63}$$

Finally, $-t^{[12]}$ just contains the term that

$$C^{[12]} \bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)} \bar{\psi}_1^{(2)} \psi_1^{(2)} \bar{\psi}_2^{(2)} \psi_2^{(2)} \bar{\psi}_1^{(3)} \psi_1^{(3)} \bar{\psi}_2^{(3)} \psi_2^{(3)}, \tag{4.64}$$

with

$C^{[12]}$

$$\begin{aligned}
&= (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_1 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_2 + 2 + iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3} \\
&+ (M_1 + 2 - iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 - g_\pi^2}{3} \\
&+ (M_2 + 2 + iH)(M_2 + 2 - iH)(M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \frac{g_\sigma^2 + g_\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
& + (M_1 + 2 + iH)(M_1 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_1 + 2 + iH)(M_2 + 2 + iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& + (M_1 + 2 + iH)(M_2 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_1 + 2 + iH)(M_3 + 2 - 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& + (M_1 + 2 + iH)(M_3 + 2 + 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_1 + 2 - iH)(M_2 + 2 + iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_1 + 2 - iH)(M_2 + 2 - iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& + (M_1 + 2 - iH)(M_3 + 2 - 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_1 + 2 - iH)(M_3 + 2 + 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& + (M_2 + 2 + iH)(M_2 + 2 - iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_2 + 2 + iH)(M_3 + 2 - 2iH) \times 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right) \\
& + (M_2 + 2 - iH)(M_3 + 2 + 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + (M_3 + 2 - 2iH)(M_3 + 2 + 2iH) \times \left[2 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^2 + \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2 \right] \\
& + 6 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right)^3 + 3 \left(\frac{g_\sigma^2 + g_\pi^2}{3} \right) \left(\frac{g_\sigma^2 - g_\pi^2}{3} \right)^2.
\end{aligned} \tag{4.65}$$

We have

$$\begin{aligned}
& - t_{i_1 j_1 k_1 l_1 m_1 n_1 i_2 j_2 k_2 l_2 m_2 n_2 i'_1 j'_1 k'_1 l'_1 m'_1 n'_1 i'_2 j'_2 k'_2 l'_2 m'_2 n'_2}^{[12]} = \\
& - C^{[12]} \delta_{j_2 + j'_1 + i_2 + i_1, 0} \delta_{i'_2 + i'_1 + j_2 + j_1, 0} \delta_{l'_2 + l'_1 + k_2 + k_1, 0} \delta_{k'_2 + k'_1 + l_2 + l_1, 0} \delta_{n'_2 + n'_1 + m_2 + m_1, 0} \delta_{m'_2 + m'_1 + n_2 + n_1, 0}.
\end{aligned} \tag{4.66}$$

4.2 Impurity tensors for $\langle |\bar{\psi} i \gamma_5 \psi|^2 \rangle$ and $\langle |\bar{\psi} i \gamma_5 \lambda_8 \psi|^2 \rangle$

These impurities introduce additional four Grassmann numbers. We set $s_{\ell, \ell'}^{(f, f')}$ tensor for $\langle \bar{\psi}_\ell^{(f)} \psi_\ell^{(f)} \bar{\psi}_{\ell'}^{(f')} \psi_{\ell'}^{(f')} \rangle$. Note that

$$s_{\ell, \ell'}^{(f, f')} = \sum_{\alpha=4,6,8,10,12} s_{\ell, \ell'}^{(f, f')[\alpha]}. \quad (4.67)$$

Impurity for $\langle |\bar{\psi} i \gamma_5 \psi|^2 \rangle$ is given by

$$s = -2 \sum_{f=1}^{N_f} s_{1,2}^{(f,f)} + 2 \sum_{f < f'} \left(s_{1,1}^{(f,f')} - s_{2,1}^{(f,f')} - s_{1,2}^{(f,f')} + s_{2,2}^{(f,f')} \right) \quad (4.68)$$

Impurity for $\langle |\bar{\psi} i \gamma_5 \lambda_8 \psi|^2 \rangle$ is given by

$$s = -\frac{2}{3} \sum_{f=1}^{N_f} s_{1,2}^{(f,f)} + \frac{2}{3} \sum_{f < f'} F_{f,f'} \left(s_{1,1}^{(f,f')} - s_{2,1}^{(f,f')} - s_{1,2}^{(f,f')} + s_{2,2}^{(f,f')} \right), \quad (4.69)$$

with

$$F_{1,2} = 1, \quad F_{1,3} = F_{2,3} = -2. \quad (4.70)$$

Let us begin with $s_{1,2}^{(1,1)}$. With $\alpha = 4$, no extra bilinear term is allowed to be entered, that is there is no extra coefficient. When $\alpha = 6$, one extra bilinear term is allowed. Since we are thinking of $s_{1,2}^{(1,1)}$, which is associated with $\bar{\psi}_1^{(1)} \psi_1^{(1)} \bar{\psi}_2^{(1)} \psi_2^{(1)}$, there are four possible terms whose coefficients correspond to $C_3^{[2]}$, $C_4^{[2]}$, $C_5^{[2]}$, and $C_6^{[2]}$. With $\alpha = 8$, two extra bilinear terms are allowed. However, they must come from the different flavor sectors (in this case, $f = 2, 3$). Hence, there are six possible terms, whose coefficients are equal to $C_9^{[4]}$, $C_{10}^{[4]}$, $C_{11}^{[4]}$, $C_{12}^{[4]}$, $C_{14}^{[4]}$, and $C_{15}^{[4]}$. For $\alpha = 10$, the room is prepared for three bilinear terms. Possible terms are four and coefficients are given by $C_{15}^{[6]}$, $C_{16}^{[6]}$, $C_{19}^{[6]}$, and $C_{20}^{[6]}$. For $\alpha = 12$, the room is just for one term, whose coefficient is $C_{15}^{[8]}$. These considerations

result in

$$\begin{aligned}
& -s_{1,2}^{(1,1)} = \\
& -\delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\bar{A}_{l'_2l'_1k_2k_1}^{(2)}A_{k'_2k'_1l_2l_1}^{(2)}\bar{A}_{n'_2n'_1m_2m_1}^{(3)}A_{m'_2m'_1n_2n_1}^{(3)} \\
& + \left(C_3^{[2]}P_{k_2k_1k'_1l_2}I_{l'_2k_2k'_2l_2} + C_4^{[2]}\right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\delta_{l'_2+l'_1+k_2+k_1,1}\delta_{k'_2+k'_1+l_2+l_1,1}\bar{A}_{n'_2n'_1m_2m_1}^{(3)}A_{m'_2m'_1n_2n_1}^{(3)} \\
& + \left(C_5^{[2]}P_{m_2m_1m'_1n_2}I_{n'_2m_2m'_2n_2} + C_6^{[2]}\right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\bar{A}_{l'_2l'_1k_2k_1}^{(2)}A_{k'_2k'_1l_2l_1}^{(2)}\delta_{n'_2+n'_1+m_2+m_1,1}\delta_{m'_2+m'_1+n_2+n_1,1} \\
& - \left(C_9^{[4]}P_{k_2k_1k'_1l_2}I_{l'_2k_2k'_2l_2}P_{m_2m_1m'_1n_2}I_{n'_2m_2m'_2n_2} + C_{10}^{[4]}P_{k_2k_1k'_1l_2}I_{l'_2k_2k'_2l_2} + C_{11}^{[4]}P_{m_2m_1m'_1n_2}I_{n'_2m_2m'_2n_2} + C_{12}^{[4]}\right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\delta_{l'_2+l'_1+k_2+k_1,1}\delta_{k'_2+k'_1+l_2+l_1,1}\delta_{n'_2+n'_1+m_2+m_1,1}\delta_{m'_2+m'_1+n_2+n_1,1} \\
& + C_{14}^{[4]}\delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\delta_{l'_2+l'_1+k_2+k_1,0}\delta_{k'_2+k'_1+l_2+l_1,0}\bar{A}_{n'_2n'_1m_2m_1}^{(3)}A_{m'_2m'_1n_2n_1}^{(3)} \\
& + C_{15}^{[4]}\delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\bar{A}_{l'_2l'_1k_2k_1}^{(2)}A_{k'_2k'_1l_2l_1}^{(2)}\delta_{n'_2+n'_1+m_2+m_1,0}\delta_{m'_2+m'_1+n_2+n_1,0} \\
& - \left(C_{15}^{[6]}P_{m_2m_1m'_1n_2}I_{n'_2m_2m'_2n_2} + C_{16}^{[6]}\right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\delta_{l'_2+l'_1+k_2+k_1,0}\delta_{k'_2+k'_1+l_2+l_1,0}\delta_{n'_2+n'_1+m_2+m_1,1}\delta_{m'_2+m'_1+n_2+n_1,1} \\
& - \left(C_{19}^{[6]}P_{k_2k_1k'_1l_2}I_{l'_2k_2k'_2l_2} + C_{20}^{[6]}\right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\delta_{l'_2+l'_1+k_2+k_1,1}\delta_{k'_2+k'_1+l_2+l_1,1}\delta_{n'_2+n'_1+m_2+m_1,0}\delta_{m'_2+m'_1+n_2+n_1,0} \\
& - C_{15}^{[8]}\delta_{j'_2+j'_1+i_2+i_1,0}\delta_{i'_2+i'_1+j_2+j_1,0}\delta_{l'_2+l'_1+k_2+k_1,0}\delta_{k'_2+k'_1+l_2+l_1,0}\delta_{n'_2+n'_1+m_2+m_1,0}\delta_{m'_2+m'_1+n_2+n_1,0}.
\end{aligned} \tag{4.71}$$

Note that $s_{1,2}^{(2,2)}$ and $s_{1,2}^{(3,3)}$ are obtained in the same way. Let us now consider flavor-mixing terms. As an example, let us focus on $s_{1,1}^{(1,2)}$. With $\alpha = 4$, no extra bilinear term is allowed to be entered, that is there is no extra coefficient. When $\alpha = 6$, one extra bilinear term is allowed. Since we are thinking of $s_{1,1}^{(1,2)}$, which is associated with $\bar{\psi}_1^{(1)}\psi_1^{(1)}\bar{\psi}_1^{(2)}\psi_1^{(2)}$, there are four possible terms whose coefficients correspond to $C_2^{[2]}$, $C_4^{[2]}$, $C_5^{[2]}$, and $C_6^{[2]}$. With $\alpha = 8$, two extra bilinear terms are allowed. There are six possible terms, whose coefficients are equal to $C_4^{[4]}$, $C_7^{[4]}$, $C_8^{[4]}$, $C_{11}^{[4]}$, $C_{12}^{[4]}$, and $C_{15}^{[4]}$. For $\alpha = 10$, the room is prepared for three bilinear terms. Possible terms are four and coefficients are given by $C_6^{[6]}$, $C_8^{[6]}$, $C_{18}^{[6]}$, and $C_{20}^{[6]}$. For $\alpha = 12$, the room is just for one term, whose coefficient is $C_{12}^{[8]}$. These

considerations result in

$$\begin{aligned}
& -s_{1,1}^{(1,2)} = \\
& P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& + C_2^{[2]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& + C_4^{[2]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& - P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \left(C_5^{[2]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_6^{[2]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& + C_4^{[4]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \bar{A}_{n'_2 n'_1 m_2 m_1}^{(3)} A_{m'_2 m'_1 n_2 n_1}^{(3)} \\
& - \left(C_7^{[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_8^{[4]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& - \left(C_{11}^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_{12}^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& - C_{15}^{[4]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \delta_{n'_2+n'_1+m_2+m_1,0} \delta_{m'_2+m'_1+n_2+n_1,0} \\
& - \left(C_6^{[6]} P_{m_2 m_1 m'_1 n_2} I_{n'_2 m_2 m'_2 n_2} + C_8^{[6]} \right) \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \delta_{n'_2+n'_1+m_2+m_1,1} \delta_{m'_2+m'_1+n_2+n_1,1} \\
& - C_{18}^{[6]} P_{k_2 k_1 k'_1 l_2} I_{l'_2 k_2 k'_2 l_2} \\
& \times \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,1} \delta_{k'_2+k'_1+l_2+l_1,1} \delta_{n'_2+n'_1+m_2+m_1,0} \delta_{m'_2+m'_1+n_2+n_1,0} \\
& - C_{20}^{[6]} P_{i_2 i_1 i'_1 j_2} I_{j'_2 i_2 i'_2 j_2} \\
& \times \delta_{j'_2+j'_1+i_2+i_1,1} \delta_{i'_2+i'_1+j_2+j_1,1} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \delta_{n'_2+n'_1+m_2+m_1,0} \delta_{m'_2+m'_1+n_2+n_1,0} \\
& - C_{12}^{[8]} \delta_{j'_2+j'_1+i_2+i_1,0} \delta_{i'_2+i'_1+j_2+j_1,0} \delta_{l'_2+l'_1+k_2+k_1,0} \delta_{k'_2+k'_1+l_2+l_1,0} \delta_{n'_2+n'_1+m_2+m_1,0} \delta_{m'_2+m'_1+n_2+n_1,0}.
\end{aligned} \tag{4.72}$$